Numerical implementation of equations for photon motion in Kerr spacetime

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ABSTRACT
Raytracing is one of the essential tools for accurate modeling of spectra and variability of various astrophysical objects. It has a major importance in relativistic environments, where light endures to a number of relativistic effects. Because the trajectories of light rays in curved spacetimes, and in Kerr spacetime in particular, are highly non-trivial, we summarize the equations governing the motion of photon (or any other zero rest mass particle) and give analytic solution of the equations that can be further used in practical computer implementations.

Keywords: Kerr spacetime – raytracing – photon motion – numerical relativity

1 INTRODUCTION

The continuously increasing sensitivity, resolution (both angular and spectral) and collecting area of telescopes and instruments onboard satellites provide astronomers with constantly better energy spectra and light curves. Those are to be compared with models of our understanding of what processes shape them, for which we need (among other things) tools that will tell us how radiation propagates from the source through a curved spacetime to a distant detector – we need so called raytracing tools.

Raytracing requires fast and accurate computations of photon trajectories through the spacetime. This can be achieved in a number of ways, but the two conceptually distinct methods use either direct numerical integration of geodesic equation or evaluation of an analytic solution. The first method is very general and can be used in any spacetime, on the other hand speed can be a limiting factor in certain applications. The second method obviously relies on the integrability of geodesics and can only be used in spacetimes, which meet this condition such as in Kerr spacetime. Its advantage is speed in case one needs to connect two distant endpoints of the trajectory or access points on the trajectory in some other way than following the path step by step with small increments. The choice between one or the other method depends mainly on the target application or the implementing code design considerations, one is not generally better than the other. Of course, raytracing is only necessary if we deal with a region of spacetime along the line of sight that hosts an extremely gravitating object. This can be a black hole or a neutron star near the the center of

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the emission or somewhere in between the source and the observer, or simply a sufficiently dense distribution of matter along that path.

This paper summarizes equations for photon trajectories in Kerr spacetime along with their analytic solutions and gives a description for their practical numerical implementation. The equations were derived by Carter (1968) but due to their complexity, a full solution was inaccessible until computers became powerful enough to do the job. Rauch and Blandford (1994) first gave the solutions to Carter's equations in terms of elliptic integrals for the Kerr metric and since then a number of authors have published their ideas about ways of solving them. Particularly good description was given by Čadež et al. (1998); Li et al. (2005); Bozza (2008); Bini et al. (2012) and Yang and Wang (2013). Different authors, however, use different approaches, symbols and different tables of integrals, so it is not always easy to compare the solutions. Moreover, most of the authors employ axial symmetry and only focus on the motion in r-θ plane.

The intention of this paper is to provide a complete description of the problem of photon motion in Kerr spacetime, summarize the original equations and provide formulae for motion in t, r, θ and φ coordinate under a unified notation. We tend to demonstrate not only solutions, but how to use them practically for a numerical implementation of a raytracing algorithm.

2 EQUATIONS OF MOTION IN CURVED SPACETIME

In general relativity, gravity is regarded as a consequence of a curved spacetime geometry, where any source of mass or energy is a source of that curvature. Trajectories of free particles copy the spacetime warp while they connect any two points along the shortest path in accord with the principle of least action. Using that principle, an equation of motion for massive and mass-less particles can be derived that is called geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\,, \tag{1}$$

where τ is an affine parameter (for a massive particle usually its proper time) and Γ is the connection tensor characterizing the shape of the spacetime. This equation can be of course solved numerically and in some trivial cases also analytically. In case of axially symmetric stationary spacetimes, the analytic solution was discovered long after the equation had been written down.

Kerr spacetime (or any stationary and axisymmetric spacetime for that matter) has two obvious symmetries that arise from the fact that its metric does not explicitly depend on time and azimuthal coordinate. This enables to find two Killing vectors associated with those differentiable symmetries that satisfy Killing equation $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$. Kerr solution also admits a hidden symmetry represented mathematically by the existence of a Killing tensor field $K'_{\mu\nu}$ – a symmetric tensor field satisfying condition $\nabla_{(\alpha}K'_{\mu\nu)} = 0$ (that the trace-free part of the symmetrization of $\nabla K'$ vanishes). This third symmetry was first appreciated by Carter (1968), who deduced the existence of an associated conserved quantity Q and demonstrated the separability of Hamilton-Jacobi equation. In addition, Kerr spacetime, as well as other $\{2, 2\}$ vacuum spacetimes, posses a conformal Killing spinor, which helps to determine parallel propagation of vectors that are perpendicular to geodesics and which leads to yet another conserved complex quantity called Walker-Penrose constant (Walker and Penrose, 1970).

According to Noether's theorem (Noether, 1918), all spacetime symmetries are related to conserved quantities. Each Killing vector corresponds to a quantity that is conserved along geodesics, meaning that the product of the Killing vector and the geodesic tangent vector is conserved along the geodesic so that $\frac{d}{d\lambda}(K_\mu \frac{dx^\mu}{d\tau}) = 0$, where τ is an affine parameter of the geodesic. The symmetry associated with Killing tensor is related to Carter's constant $Q = K'_{\mu\nu} u^\mu u^\nu$. Physically, these three constants correspond to the conserved energy, the angular momentum with respect to the symmetry axis of the black hole, and the square of the total angular momentum along the geodesic (Bardeen et al., 1972; Wald, 1984). Beside this, the norm of four-velocity thanks to its parallel propagation is a fourth conserved number.

Carter's discovery of conserved quantity Q allowed him to explicitly demonstrate the separability of Hamilton-Jacobi equation and enabled to solve general geodesic motion in Kerr spacetime analytically (Misner et al., 1973; Chandrasekhar, 1983). Following his approach, the Hamilton-Jacobi equation for the Hamilton's principal function S,

$$2\frac{\partial S}{\partial \tau} = g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} \,, \tag{2}$$

becomes after evaluation of metric coefficients

$$\begin{aligned} 2\frac{\partial S}{\partial \tau} =& \frac{1}{\rho^2 \Delta} \left[(r^2+a^2)\frac{\partial S}{\partial t} + a\frac{\partial S}{\partial \varphi} \right]^2 - \frac{1}{\rho^2 \sin^2\theta} \left[a\sin\theta \frac{\partial S}{\partial t} + \frac{\partial S}{\partial \varphi} \right]^2 \\ & - \frac{\Delta}{\rho^2} \left(\frac{\partial S}{\partial r} \right)^2 - \frac{1}{\rho^2} \left(\frac{\partial S}{\partial \theta} \right)^2 \end{aligned} \tag{3}$$

and we seek a solution in a form

$$S = \tfrac{1}{2}\delta\tau - Et + L_z\varphi + S_r(r) + S_\theta(\theta) \,, \tag{4}$$

where the negative sign of energy or the factor $1/2$ is chosen for convenience and already anticipates the final solution.

When the Hamilton-Jacobi equation (3) is applied to the anzatz (4) it turns out that the resulting equation can be arranged in such a way that on one side it has terms that only contain explicit dependence on r coordinate and on other side it has terms with θ dependence. Using a separation constant Q, it can thus be split into two equations

$$\left(\frac{dS_r}{dr} \right)^2 = \frac{R(r)}{\Delta^2} \quad \text{and} \quad \left(\frac{dS_\theta}{d\theta} \right)^2 = \Theta(\theta) \,, \tag{5}$$

where

$$R(r) = [(r^2+a^2)E - aL_z]^2 - \Delta[Q + (L_z - aE)^2 + \delta r^2] \,, \tag{6}$$

$$\Theta(\theta) = Q - (L_z^2 \sin^{-2}\theta - a^2E^2 + \delta a^2)\cos^2\theta \tag{7}$$

and Q is Carter's separation constant. Details can be found in Chandrasekhar (1983), Chapter 62.

The solution for the principal function S then becomes

$$S = \tfrac{1}{2}\delta\tau - Et + L_z\varphi + \int^r \frac{\sqrt{R(r)}}{\Delta}\,\mathrm{d}r + \int^\theta \sqrt{\Theta(\theta)}\,\mathrm{d}\theta\,. \tag{8}$$

and the basic equations for geodesic motion can be obtained from the solution by calculating the partial derivatives of S with respect to different constants of motion and setting those to zero. Starting with Carter's constant, we find that

$$\frac{\partial S}{\partial Q} = \frac{1}{2}\int \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial Q}\,\mathrm{d}r + \frac{1}{2}\int \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial Q}\,\mathrm{d}\theta = 0 \tag{9}$$

that leads to

$$\int^r \frac{1}{\sqrt{R}}\,\mathrm{d}r = \int^\theta \frac{1}{\sqrt{\Theta}}\,\mathrm{d}\theta\,. \tag{10}$$

This frequently used equation describes the geodesic trajectory in $[r,\theta]$ plane, which provides sufficient description of the trajectory shape in applications that have stationary and axially symmetric geometry. For instance, many raytracing applications only use this equation for computing, e.g., an accretion disk image or spectrum, if the disk is assumed to be axially symmetric.

Similarly, we find a relation for photon's affine parameter (what would be the proper time for massive particles)

$$\begin{aligned}\frac{\partial S}{\partial \delta} &= \frac{1}{2}\tau + \frac{1}{2}\int \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial \delta}\,\mathrm{d}r + \frac{1}{2}\int \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial \delta}\,\mathrm{d}\theta \\ &= \frac{1}{2}\tau + \frac{1}{2}\int \frac{r^2}{\sqrt{R}}\,\mathrm{d}r + \frac{1}{2}\int \frac{a^2\cos^2\theta}{\sqrt{\Theta}}\,\mathrm{d}\theta = 0\,,\end{aligned} \tag{11}$$

$$\tau = \int^r \frac{r^2}{\sqrt{R}}\,\mathrm{d}r + a^2\int^\theta \frac{\cos^2\theta}{\sqrt{\Theta}}\,\mathrm{d}\theta\,. \tag{12}$$

For energy E as a constant of motion we have

$$\begin{aligned}\frac{\partial S}{\partial E} &= -\,t + \frac{1}{2}\int \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial E}\,\mathrm{d}r + \frac{1}{2}\int \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial E}\,\mathrm{d}\theta \\ &= -\,t + \int \frac{r^4E + r^2a^2E + 2Ma^2rE - 2MarL_z}{\Delta\sqrt{R}}\,\mathrm{d}r\,+ \\ &\quad + \int \frac{Ea^2\cos^2\theta}{\sqrt{\Theta}}\,\mathrm{d}\theta = 0\,,\end{aligned} \tag{13}$$

which can be expressed after some rearranging with the help of (12) as

$$t = \tau E + 2M\int^r \frac{r^3E - a(L_z - aE)r}{\Delta\sqrt{R}}\,\mathrm{d}r\,. \tag{14}$$

And finally, the derivative of S with respect to L_z gives

$$\begin{aligned}\frac{\partial S}{\partial L_z} &= \varphi + \frac{1}{2}\int \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial L_z}\,\mathrm{d}r + \frac{1}{2}\int \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial L_z}\,\mathrm{d}\theta \\ &= \varphi - \int \frac{2MrL_z - 2MarE - r^2L_z}{\Delta\sqrt{R}}\,\mathrm{d}r - \int \frac{L_z\cot^2\theta}{\sqrt{\Theta}}\,\mathrm{d}\theta = 0\,,\end{aligned} \tag{15}$$

$$\varphi = \int^r \frac{r(r-2M)L_z + 2MarE}{\Delta\sqrt{R}}\,\mathrm{d}r + \int^\theta \frac{L_z\cot^2\theta}{\sqrt{\Theta}}\,\mathrm{d}\theta\,, \tag{16}$$

which can be proved to be the same formula as the one given by Chandrasekhar (1983, Equation 181)

$$\varphi = a\int^r \frac{(r^2+a^2)E - aL_z}{\Delta\sqrt{R}}\,\mathrm{d}r + \int^\theta \frac{L_z\sin^{-2}\theta - aE}{\sqrt{\Theta}}\,\mathrm{d}\theta\,. \tag{17}$$

3 SOLUTIONS FOR PHOTON TRAJECTORIES

Null geodesics have $\delta = 0$ and if we define dimensionless quantities for the constant of motions $\lambda = L_z/E = -p_\varphi/p_t$ and $q = Q/E^2$ and replace $\cos\theta$ with μ, the set of equations of motion become

$$\int^r \frac{1}{\sqrt{R(r)}}\,\mathrm{d}r = \int^\mu \frac{1}{\sqrt{\Theta_\mu(\mu)}}\,\mathrm{d}\mu\,, \tag{18}$$

$$\Delta t = \int^r \frac{r^2(r^2+a^2) + 2ar(a-\lambda)}{\Delta\sqrt{R(r)}}\,dr + a^2\int^\mu \frac{\mu^2}{\sqrt{\Theta_\mu(\mu)}}\,\mathrm{d}\mu\,, \tag{19}$$

$$\Delta\phi = \int^r \frac{2ar - \lambda a^2}{\Delta\sqrt{R(r)}}\,\mathrm{d}r + \int^\mu \frac{\lambda}{(1-\mu^2)\sqrt{\Theta_\mu(\mu)}}\,\mathrm{d}\mu\,, \tag{20}$$

$$R(r) = r^4 + (a^2 - \lambda^2 - q)r^2 + 2\left(q + (\lambda - a)^2\right)r - a^2q\,, \tag{21}$$

$$\Theta_\mu(\mu) = q + (a^2 - \lambda^2 - q)\mu^2 - a^2\mu^4 = a^2(\mu_-^2 + \mu^2))(\mu_+^2 - \mu^2))\,. \tag{22}$$

This should be accompanied by equations for photon 4-momentum

$$\Sigma\frac{\mathrm{d}k^t}{\mathrm{d}\tau} = -a\left[a(1-\mu^2) - \lambda\right] + (r^2+a^2)\frac{r^2+a^2-a\lambda}{\Delta}\,, \tag{23a}$$

$$\Sigma\frac{\mathrm{d}k^r}{\mathrm{d}\tau} = \pm\sqrt{R(r)}\,, \tag{23b}$$

$$\Sigma\frac{\mathrm{d}k^\theta}{\mathrm{d}\tau} = \pm\sqrt{\Theta(\theta)} = \pm\sqrt{\Theta_\mu(\mu)/(1-\mu^2)}\,, \tag{23c}$$

$$\Sigma\frac{\mathrm{d}k^\varphi}{\mathrm{d}\tau} = -a + \frac{\lambda}{1-\mu^2} + \frac{a(r^2+a^2-a\lambda)}{\Delta}\,. \tag{23d}$$

The integration over r and μ goes from one endpoint (an emitter) to the other (an observer) of the trajectory and the signs of $R^{1/2}$ and $\Theta_\mu^{1/2}$ must be the same as those of $\mathrm{d}r$ and $\mathrm{d}\theta$, which change each time the integration passes through a radial or a poloidal turning point, where $R(r)=0$ or $\Theta_\mu(\mu)=0$, respectively. There can be at most one radial turning point, but many poloidal turning points along a particular trajectory as it may wind up around the photon orbit.

Starting with Eq. 18, both $R(r)$ and $\Theta_\mu(\mu)$ are quartic polynomials, meaning that both integrals can be evaluated in terms of an elliptic integral of the first kind $F(\phi|m)$, where the modulus is only function of the constants of motion and ϕ is a suitable function of r or μ, respectively. Calculating the change in the azimuthal position and the travel time involves more complex integrands that evaluate to elliptic integral of the second and third kind.

Eventually, all necessary integrals can be expressed in terms of Jacobi elliptic functions $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, $\mathrm{dn}(u|m)$, $\mathrm{tn}(u|m)$ and their inversions and in terms of elliptic integrals of the first, second and third kind that boil down to evaluation of Carlson elliptic integral functions. The book of elliptic integrals by Byrd and Friedman (1971, hereafter BF) becomes helpful in finding the transformations of various integrals to Jacobi and Carlson functions. Numerical implementation of Carlson's elliptic integrals $\mathrm{R_C}(x,y)$, $\mathrm{R_D}(x,y,z)$, $\mathrm{R_F}(x,y,z)$, $\mathrm{R_J}(x,y,z,p)$, Legendre elliptic integrals $\mathrm{F}(\phi|m)$, $\mathrm{E}(\phi|m)$, $\Pi(\phi|m)$ and Jacobi elliptic functions $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, $\mathrm{dn}(u|m)$, $\mathrm{tn}(u|m)=\mathrm{sn}(u|m)/\mathrm{cn}(u|m)$ can be found e.g. in Press (1992, Numerical Recipes, Chapter 6). For inverse Jacobi elliptic functions, we can use the relation

$$z=\mathrm{sn}(u|m)\,,\quad u=\mathrm{sn}^{-1}(z|m) \tag{24}$$

and similarly for $\mathrm{cn}(u|m)$, $\mathrm{dn}(u|m)$ and $\mathrm{tn}(u|m)$. Then we get from the definition of the functions (BF, Eq. 131.00)

$$\mathrm{sn}^{-1}(z|m)=z\ \mathrm{R_F}\left(1-z^2,1-mz^2,1\right), \tag{25a}$$
$$\mathrm{cn}^{-1}(z|m)=\mathrm{sn}^{-1}(\sqrt{(1-z^2)}\,|\,m), \tag{25b}$$
$$\mathrm{dn}^{-1}(z|m)=\mathrm{sn}^{-1}(\sqrt{(1-z^2)/m}\,|\,m), \tag{25c}$$
$$\mathrm{tn}^{-1}(z|m)=\mathrm{sn}^{-1}(\sqrt{z^2/(1+z^2)}\,|\,m), \tag{25d}$$

A convention for the modulus in elliptic integrals and elliptic functions is used throughout this paper, which differs by a square from what Byrd and Friedman (1971) use; e.g. what we denote as $\mathrm{sn}(u|m)$ is $\mathrm{sn}(u|\sqrt{m})$ in BF.

3.1 Radial integral

In order to express the R-integral in terms of elliptic functions, we need to factorize function $R(r)$ and find its roots r_1, r_2, r_3, r_4. Since $R(0)=-a^2q$, there are two options. For geodesics that cross the equatorial plane and have $q\geq 0$ or $a=0$, $R(0)\leq 0$ and because $R(\pm\infty)\to+\infty$, the expression has to have two or four real roots. For geodesics that have $q<0$ and do not cross the equatorial plane, $R(0)>0$ and $R(r)$ can have four or zero real roots. We can thus consider three separate cases:

Four real roots. Let us assume roots in the order $r_1 > r_2 > r_3 > r_4$, then $R(0) = r_1\,r_2\,r_3\,r_4 = -a^2 q$ means that $r_4 < 0$ and that photons are allowed to exist in region $r > r_1$ (outer region, both endpoints at infinity) or in region $r_3 < r < r_2$ (inner region, both endpoints under the event horizon). Analytic formulae for evaluating the root values have been given by Čadež et al. (1998). In the outer region, the integral evaluates to (BF, Eq. 258.00)

$$\int_{r_1}^{r} \frac{\mathrm{d}r}{\sqrt{R(r)}} = \int_{r_1}^{r} \frac{\mathrm{d}r}{\sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}} \\ = \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}}\,\mathrm{sn}^{-1}\left[\sqrt{\frac{(r_2-r_4)(r-r_1)}{(r_1-r_4)(r-r_2)}}\,\middle|\, m_4\right], \tag{26}$$

where

$$m_4 = \frac{(r_1-r_4)(r_2-r_3)}{(r_1-r_3)(r_2-r_4)}\,. \tag{27}$$

In the inner region, the integral evaluates to (BF, Eq. 255.00)

$$\int_{r}^{r_2} \frac{\mathrm{d}r}{\sqrt{R(r)}} = \int_{r}^{r_2} \frac{\mathrm{d}r}{\sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}} \\ = \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}}\,\mathrm{sn}^{-1}\left[\sqrt{\frac{(r_1-r_3)(r_2-r)}{(r_2-r_3)(r_1-r)}}\,\middle|\, m_4\right]. \tag{28}$$

In case of double real root $r_1 = r_2$, the above expressions fail, because the inverse Jacobi elliptic function $\mathrm{sn}^{-1}(1|1) = \infty$ and the evaluation requires a special treatment (Li et al., 2005, Eq. A11 and A13). In practice, however, such a case hardly ever happens with a numerical approach.

Two real and two complex roots. Let us assume roots in the order $r_1 > r_2$ for the two real roots and $r_3 = r_4^\star$ (complex conjugate) for the two complex roots. Since $r_1 r_2 r_3 r_4 = r_1 r_2 |r_3|^2 = -a^2 Q \le 0$, r_1 must be positive and r_2 must be negative. They cannot be equal as in that case they both would have to be zero, for which (21) implies also $r_3 = r_4 = 0$. The R-integral then evaluates to (BF, Eq. 260.00)

$$\int_{r_1}^{r} \frac{\mathrm{d}r}{\sqrt{R(r)}} = \frac{1}{\sqrt{AB}}\,\mathrm{cn}^{-1}\left[\frac{(A-B)r + r_1 B - r_2 A}{(A+B)r - r_1 B - r_2 A}\,\middle|\, m_2\right]. \tag{29}$$

where

$$m_2 = \frac{(A+B)^2 - (r_1-r_2)^2}{4AB}\,, \tag{30a}$$

$$A = \left[(r_1-u)^2 + v^2\right]^{1/2}, \quad B = \left[(r_2-u)^2 + v^2\right]^{1/2} \tag{30b}$$

with u and v being respectively the real and imaginary part of r_3.

Four complex roots roots. Let us assume roots $r_1 = r_2^\star$ and $r_3 = r_4^\star$ (* stands for a complex conjugate). Then $R(r)$ can be written as $R(r) = [(r-b_1)^2 + a_1^2][(r-b_2)^2 + a_2^2]$ and the radial integral is (BF, Eq. 267.00)

$$\int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{R(r)}} = \int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{[(r-b_1)^2 + a_1^2][(r-b_2)^2 + a_2^2]}} = \\ = \frac{2}{A+B}\,\mathrm{tn}^{-1}\left(\frac{r-r_0}{a_1 + b_1 g_1 - g_1 r}\,\middle|\, m_0\right), \tag{31}$$

where

$$b_1 = (r_1+r_2)/2 = \mathrm{Re}(r_1), \qquad a_1^2 = -(r_1-r_2)^2/4 = \mathrm{Im}(r_1)^2, \tag{32a}$$
$$b_2 = (r_3+r_4)/2 = \mathrm{Re}(r_3), \qquad a_2^2 = -(r_3-r_4)^2/4 = \mathrm{Im}(r_3)^2, \tag{32b}$$
$$A^2 = (b_1-b_2)^2 + (a_1+a_2)^2, \qquad B^2 = (b_1-b_2)^2 + (a_1-a_2)^2, \tag{32c}$$
$$g_1 = \frac{4a_1^2 - (A-B)^2}{(A+B)^2 - 4a_a^2}, \qquad m_0 = \frac{4AB}{(A+B)^2}\,. \tag{32d}$$

Note that now the lower limit of the integral does not go from the largest real root of $R(r)$, but it is

$$r_0 = b_1 - a_1 g_1\,. \tag{33}$$

Let us introduce a quantity

$$P_r(r) \equiv \int_r^\infty \frac{\mathrm{d}r}{\sqrt{R(r)}} = \int_{r_1}^\infty \frac{\mathrm{d}r}{\sqrt{R(r)}} \pm \int_{r_1}^{r} \frac{\mathrm{d}r}{\sqrt{R(r)}} = P_{r_1} \pm \int_{r_1}^{r} \frac{\mathrm{d}r}{\sqrt{R(r)}}\,. \tag{34}$$

We can then describe the whole trajectory by the value of P_r, which is zero for $r=\infty$, then increases up to $P_{r_1} = \int_{r_1}^\infty$ as r reaches the turning point and further increases to $P_r = 2P_{r_1}$ as $r \to \infty$ at the other end of the trajectory (behind the pericenter). Such a parametrization can be used instead of the affine parameter (if we do not need the affine property) as the value of P_r uniquely identifies any point of the trajectory. For the inbound orbit this is similarly

$$P_r(r) \equiv \int_{r_3}^{r} \frac{\mathrm{d}r}{\sqrt{R(r)}} = \int_{r_3}^{r_2} \frac{\mathrm{d}r}{\sqrt{R(r)}} \pm \int_{r}^{r_2} \frac{\mathrm{d}r}{\sqrt{R(r)}} = P_{r_2} \pm \int_{r}^{r_2} \frac{\mathrm{d}r}{\sqrt{R(r)}}\,. \tag{35}$$

Equations (26), (28), (29) and (31) can be inverted based on relations (24) and (25) to obtain radius from the known value of the integral.

For the case of four real roots and the outer region we have

$$r = \frac{r_1(r_2-r_4) - r_2(r_1-r_4)\,\mathrm{sn}^2(\xi_{4a}|m_4)}{(r_2-r_4) - (r_1-r_4)\,\mathrm{sn}^2(\xi_{4a}|m_4)}\,, \tag{36}$$

where

$$\xi_{4a} = \pm\frac{1}{2}(P_r - P_{r_1})\sqrt{(r_1 - r_3)(r_2 - r_4)} \tag{37}$$

with the sign chosen such that ξ_{4a} is positive.

For the case of four real roots and the inner region we have

$$r = \frac{r_2(r_1 - r_3) - r_1(r_2 - r_3)\,\mathrm{sn}^2(\xi_{4b}|m_4)}{(r_1 - r_3) - (r_2 - r_3)\,\mathrm{sn}^2(\xi_{4b}|m_4)}\,, \tag{38}$$

where

$$\xi_{4b} = \pm\frac{1}{2}(P_r - P_{r_2})\sqrt{(r_1 - r_3)(r_2 - r_4)}\,. \tag{39}$$

For the case of two real and two complex roots we have

$$r = \frac{(r_2 A - r_1 B) - (r_2 A + r_1 B)\,\mathrm{cn}(\xi_2|m_2)}{(A - B) - (A + B)\,\mathrm{cn}(\xi_2|m_2)}\,, \tag{40}$$

where $\xi_2 = (P_{r_1} - P_r)\sqrt{AB}$ and A and B are given by (30).
Finally, for the case of four complex roots we have

$$r = \frac{(b_1 - a_1 g_1) + (a_1 + b_1 g_1)\,\mathrm{tn}(\xi_0|m_0)}{1 + \mathrm{tn}(\xi_0|m_0)}\,, \tag{41}$$

where $\xi_0 = (P_{r_0} - P_r)(A + B)/2$ and r_0, A, B are given by (32).

3.2 Poloidal integral

Evaluation of the poloidal integral also depends on the value of q. From (22) follows that $\mu_+^2\mu_-^2 = q/a^2$. If $q > 0$, both μ_+^2 and μ_-^2 must be positive and the poloidal motion is allowed in the range of $-\mu_+ < \mu < +\mu_+$. If $q < 0$, μ_-^2 must be negative (which causes no problems) and the motion is allowed only in the range of $\sqrt{-\mu_-^2} < \mu < \mu_+$ and $-\mu_+ < \mu < -\sqrt{-\mu_-^2}$, and such orbits do not cross the equatorial plane. Moreover, $q + (a - \lambda)^2$ must be positive. Based on these two cases, the integral on the right-hand side of (18) can be worked out as (BF, Eq. 213.00 and 218.00)

$$\int_{\mu}^{\mu_+} \frac{\mathrm{d}\mu}{\sqrt{\Theta_\mu(\mu)}} = \int_{\mu}^{\mu_+} \frac{\mathrm{d}\mu}{\sqrt{a^2(\mu_-^2 + \mu^2)(\mu_+^2 - \mu^2)}} =$$

$$= \begin{cases} \dfrac{1}{\sqrt{a^2(\mu_+^2 + \mu_-^2)}}\,\mathrm{cn}^{-1}\left(\frac{\mu}{\mu_+}\middle|\,\frac{\mu_+^2}{\mu_+^2 + \mu_-^2}\right) & \text{for } \mu_-^2 > 0\,, \\[2ex] \dfrac{1}{\sqrt{a^2\mu_+^2}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\mu_+^2 - \mu^2}{\mu_+ + \mu_-}}\,\middle|\,\frac{\mu_+^2 + \mu_-^2}{\mu_+^2}\right) & \text{for } \mu_-^2 < 0\,. \end{cases} \tag{42}$$

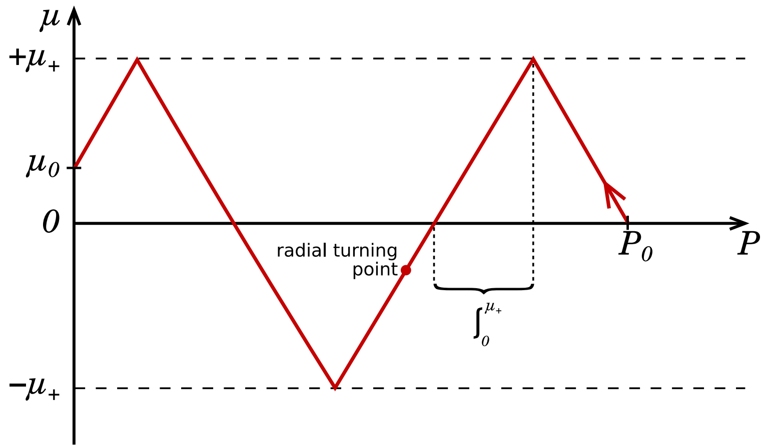

Figure 1. An illustration of a photon trajectory shown in terms of varying poloidal coordinate $\mu = \cos\theta$ (vertical axis) and radial coordinate (horizontal axis). The radial coordinate is, however, expressed indirectly by the value of the R-integral. Note that the radial and poloidal integral change at the same pace ($P_r(r) = P_\mu(\mu)$), so the path has a triangular shape and segments between any two turning poloidal points have identical change in the value of the integral.

It should be noted that trajectories may oscillate between turning points several times depending on the exact aiming. For example (see Fig. 1 and read it from right to left), we may have a trajectory that starts off the equatorial plane, goes up, then crosses the equatorial plane $\mu = 0$, goes bellow it, then back above it and it finally reaches its final value of poloidal angle μ_0 after passing through one more turning point at μ_+. The value of P is zero at the end of trajectory, which means that this end is at infinity, while it starts from the equatorial plane at some radius r_0, which corresponds to the value of the radial integral $P = P_0$. Somewhere along the trajectory, there lies also the radial turning point.

For the trajectory in Fig. 1, the value of the poloidal integral is

$$P_\mu = \left(\int_0^{+\mu_+} + 2\int_{-\mu_+}^{+\mu_+} + \int_{\mu_0}^{+\mu_+} \right) \frac{\mathrm{d}\mu}{\sqrt{\Theta_\mu(\mu)}} \tag{43}$$

and it has to match the value of the radial integral at each point of the trajectory,

$$P_r(r) = P_\mu(\mu)\,, \tag{44}$$

which makes a unique link between radial and poloidal coordinate at any given point along the trajectory and allows, from a known value P of the radial/poloidal integral, to determine coordinates $r(P)$ (Eq. 36, 38, 40, 41) and $\mu(P)$ through inversion

$$\mu = \mu_+ \,\mathrm{cn}\left(\frac{P_\mu}{m_K} \middle| m_\mu \right) . \tag{45}$$

The last expression requires some caution, because it only works within a single segment of the trajectory between $-\mu_+$ and $+\mu_+$ meaning that P_μ has to be in the range $0 \le P_\mu \le$

$2P_\mu(0)$ and a suitable multiple of $\int_0^{\mu_+}$ has to be subtracted from P_μ together with any bit of $\int_{\mu_0}^{\mu_+}$ remaining at the other end of the trajectory.

3.3 Time integral

The time integral (19) can be solved after the polynomial in the radial part of the integral has been factorized. For the integrand of the radial part we get (dropping factor $1/\sqrt{R(r)}$ for a moment)

$$\frac{r^2(r^2+a^2)+2ar(a-\lambda)}{\Delta} = r^2+2r+4+\frac{2(-a\lambda+4)r-4a^2}{\Delta}\,. \tag{46}$$

The first three terms can be evaluated with suitable integrals and we can still manipulate with the fourth term assuming that $\Delta = r^2-2r+a^2 = (r-r_+)(r-r_-)$. After bit of an algebra, we find that

$$\frac{2(-a\lambda+4)r-4a^2}{\Delta} = \frac{2(-a\lambda+4)r_+-4a^2}{(r_+-r_-)(r-r_+)} + \frac{-2(-a\lambda+4)r_++4a^2}{(r_+-r_-)(r-r_-)}\,. \tag{47}$$

The final expression for coordinate travel time therefore is

$$\begin{aligned}
\Delta t = & \int^r \frac{r^2}{\sqrt{R(r)}}\,\mathrm{d}r + 2\int^r \frac{r}{\sqrt{R(r)}}\,\mathrm{d}r + 4\int^r \frac{1}{\sqrt{R(r)}}\,\mathrm{d}r + \\
& + \frac{(-a\lambda+4)r_+-2a^2}{\sqrt{1-a^2}}\int^r \frac{1}{(r-r_+)\sqrt{R(r)}}\,\mathrm{d}r - \\
& - \frac{(-a\lambda+4)r_--2a^2}{\sqrt{1-a^2}}\int^r \frac{1}{(r-r_-)\sqrt{R(r)}}\,\mathrm{d}r + \\
& + a^2\int^\mu \frac{\mu^2}{\sqrt{\Theta_\mu(\mu)}}\,\mathrm{d}\mu\,.
\end{aligned} \tag{48}$$

The integrals in the above expression can all be evaluated in terms of elliptic functions. The expressions get lengthy, so we only point out the formulae to be used. In case of 4 real roots of $R(r)$ function, we use BF formula 258.11+340.02 for the first integral, 258.11+340.01 for the second, 258.00 for the third and 258.39+340.01 for the last two radial integrals. In case of 2 real and 2 complex roots of $R(r)$, we use formulae 260.03+341.02-04, 260.03+341.02-03, 260.00 and 260.04+341.02-03, respectively. In case of 4 complex roots, we use formulae 267.01+342.02-04, 267.01+341.02-03, 267.00 and 267.02+342.02-03, respectively. The poloidal integral can be evaluated using formula 213.06+312.02 or 218.01. All integrals have to be taken with proper limits as it has been discussed in Sections 3.1 and 3.2.

3.4 Azimuthal integral

Finally, the change of azimuthal coordinate along the trajectory can be calculated from Eq. 20. After a similar factorization of the radial part of the integral that we have used in the previous section we obtain

$$
\begin{aligned}
\Delta\varphi = &\int^{r} \frac{2ar - \lambda a^2}{\Delta\sqrt{R(r)}}\, \mathrm{d}r + \int^{\mu} \frac{\lambda}{(1-\mu^2)\sqrt{\Theta_\mu(\mu)}}\, \mathrm{d}\mu = \\
= &\frac{2ar_+ - \lambda a^2}{2\sqrt{1-a^2}} \int^{r} \frac{\mathrm{d}r}{(r-r_+)\sqrt{R(r)}} - \frac{2ar_- - \lambda a^2}{2\sqrt{1-a^2}} \int^{r} \frac{\mathrm{d}r}{(r-r_-)\sqrt{R(r)}} + \\
&+ \lambda \int^{\mu} \frac{\mathrm{d}\mu}{(1-\mu^2)\sqrt{\Theta_\mu(\mu)}}\,. \qquad (49)
\end{aligned}
$$

The first two radial integrals are of the same type as the terms in the time integral and the same formulae as listed in the previous subsection are used to evaluate them. The last integral can be evaluated using formula 213.02 or 218.02. Again, all integrals has to be taken with proper limits.

3.5 Integration of geodesic equation

We spend the rest of this section with another way of getting a purely numerical solution of photon trajectories that directly integrates the geodesic equation (1). Such an approach becomes useful when one deals with a non-Kerr metric, either a different analytic metric or a numeric metric. In fact, it is often the only possible approach in such cases.

There are many methods published in the literature that can be used to integrate second-order differential equations with Runge-Kutta method probably being in the lead of the most frequently used ones. Other choices my include Dormand-Prince method or Burlirsch-Stoer algorithm. Those methods however require multiple evaluation of the derivative function, which can be time consuming. This is indeed the case of geodesic equation, where components of the connection tensor have to be evaluated. This is rank 3 tensor that may contain up to 40 independent components. If performance is an issue, one seeks for a method that gives a reasonable precision while minimizing the number of necessary evaluation of the derivatives. Verlet algorithm is one of such methods.

Verlet algorithm is a numerical method frequently used to integrate equations of motion in Newtonian kinematics or particle trajectories in molecular dynamics simulations. Dolence et al. (2009) give a version of the algorithm for integrating the geodesic equation. The integration starts with an initial position x_0^μ of the photon, its initial 4-momentum k_0^μ

and momentum derivative $\mathrm{d}k_0^\mu/\,\mathrm{d}\lambda$ and follows the scheme

$$x^\mu_{n+1} = x^\mu_n + k^\mu_n \Delta\lambda + \frac{1}{2}\left(\frac{\mathrm{d}k^\mu}{\mathrm{d}\lambda}\right)_n (\Delta\lambda)^2\,, \tag{50a}$$

$$k'^\mu = k^\mu_n + \left(\frac{\mathrm{d}k^\mu}{\mathrm{d}\lambda}\right)_n \Delta\lambda\,, \tag{50b}$$

$$\left(\frac{\mathrm{d}k^\mu}{\mathrm{d}\lambda}\right)_{n+1} = -\Gamma^\mu_{\alpha\beta}(x_{n+1})\, k'^\alpha\, k'^\beta\,, \tag{50c}$$

$$k^\mu_{n+1} = k^\mu_n + \frac{1}{2}\left[\left(\frac{\mathrm{d}k^\mu}{\mathrm{d}\lambda}\right)_n + \left(\frac{\mathrm{d}k^\mu}{\mathrm{d}\lambda}\right)_{n+1}\right]\Delta\lambda\,. \tag{50d}$$

(50e)

Intermediate steps (50b) and (50c) are repeated few times until the required accuracy is reached. The expensive calculation of the connection coefficients Γ is, however, done only once. The accuracy can be controlled at each step by checking the error of $k^\mu k_\mu$ (that is supposed to be zero), k_t or Carter's constant Q (that are supposed to conserve their initial values) and the integration routine can adjust the step size $\Delta\lambda$ accordingly.

4 NUMERICAL IMPLEMENTATION

All the above given formulae are implemented in SIM5[1] library, which is an extensive collection of routines for raytracing and radiative transport in general relativity.

The core of the library contains a C implementation of Carlson elliptic integrals and Jacobi elliptic functions after Press (1992), with various enhancements. Those function have been checked against their implementation in Mathematica, ver. 7 to provide same results including extensions to the supported range of parameters where necessary. This is supplemented by implementation of required formulae for reduction of elliptic integrals from Byrd and Friedman (1971) book.

On top of these low-level functions, the library builds a set of routines that give solutions for integrals of photon motions. Those routines use a uniform way of parametrization of geodesics based on the value of R-integral.

There is also a routine for numerical integration of the geodesic equation, which is an implementation of Eq. 50. The routine controls the accuracy of the integration by checking the values of $k^\mu k_\mu$ and k_t, which can be evaluated faster than Carter's constant, and adjusts the step size $\Delta\lambda$ to keep the accuracy in tolerance. The overall precision of the integration is controlled via a parameter, allowing for faster less accurate or slower and more reliable calculations.

The library is freely available for use at GitHub.

[1] `https://github.com/mbursa/sim5`

5 CONCLUSIONS

The paper outlines ways of solving photon geodesics (mainly) in Kerr spacetime with the aim of giving a complete summary of the equations needed for obtaining all the coordinates along a trajectory. Major focus is put on the analytic solution of the equations of photon motion using elliptic integrals and elliptic functions, because it provides a fast and efficient way of evaluating light ray paths in Kerr spacetime. Sometimes, a step-wise approach of geodesic integration may be better suited for a particular task or a different spacetime than Kerr is of interest. For those cases, a method of direct numerical integration of geodesic equation is discussed, which is one of many ways how such an equation may be integrated, however it is a choice that gives balanced ratio between performance and accuracy.

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