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Magnetic field generated by current loop in flat spacetime

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ABSTRACT
Magnetic field generated by circular current loop located in flat spacetime is studied. Ampere's law has been solved using two different methods of expansions into infinite series: multipole expansion and power series expansion. Both infinite series solutions are compared with full analytic solution and their pros and cons are discusses.

Keywords: current loop – magnetic field – flat spacetime

1 INTRODUCTION

Gravitation and electromagnetisms are the only two long range forces important in astrophysics. Gravitational collapsed object - black hole can be well described by Kerr metric if we will assume electromagnetic field to be weak, i.e. stress-energy tensor for electromagnetic field does not contribute to the geometry of background spacetime. Any electric charge will be discharged quite quick for astrophysical compact object surrounded by plasma, so only magnetic field will remain in play. The exact shape and intensity of magnetic field surrounding the black hole (black hole magnetosphere) is still not yet properly resolved, but strong connection to the accretion processes is assumed (Meier, 2012). In this article we will focus on magnetic field generated by circular current loop, located in equatorial plane, as a model of toroidal current floating inside the accretion disks. We will focus on flat spacetime in this preliminary study only, but full relativistic approach will be used.

Our search for proper shape of black hole magnetosphere is motivated by our study of charged particle motion in vicinity of magnetized black hole (Kološ et al., 2015; Stuchlík and Kološ, 2016) where we have been using simple model of uniform magnetic field only, and now we are looking for more realistic magnetic field solution. It is known, that even weak magnetic field can have significant influence on the charged particles motion - all depend on test particle specific charge (particle charge to mass ratio). The "charged particle" can represent matter ranging from electron to some charged inhomogeneity orbiting in the innermost region of the accretion disk surrounding the black hole. To have a smooth

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charged particle trajectory, the spacetime metric and tensor of electromagnetic field must be sufficiently smooth in the region where the motion of particle occur. As we will see further, some magnetic field solutions are composed of two branches matched at some radii and hence does not meet smoothness criteria.

2 MAXWELL'S EQUATIONS

Gravity will not be included at this stage and the spacetime is flat with line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{1}$$

Cartesian coordinates x, y, z can be obtained by the coordinate transformations

$$x = r\,\cos\phi\,\sin\theta, \quad y = r\,\sin\phi\,\sin\theta, \quad z = r\,\cos\theta. \tag{2}$$

Vectors and tensors can be expressed in local frame of reference using relations

$$A^{\widehat{\mu}} = \frac{\partial x^{\widehat{\mu}}}{\partial x^{\mu}} A^{\mu}, \qquad F^{\widehat{\mu\nu}} = \frac{\partial x^{\widehat{\mu}}}{\partial x^{\mu}} \frac{\partial x^{\widehat{\nu}}}{\partial x^{\nu}} F^{\mu\nu}, \tag{3}$$

where the local observer coordinates are

$$\mathrm{d}\widehat{t} = \mathrm{d}t, \quad \mathrm{d}\widehat{r} = \mathrm{d}r, \quad \mathrm{d}\widehat{\theta} = r\,\mathrm{d}\theta, \quad \mathrm{d}\widehat{\phi} = r\sin\theta\,\mathrm{d}\phi. \tag{4}$$

The metric $g_{\widehat{\mu\nu}}$ constructed by local observer coordinates has very simple form $g_{\widehat{\mu\nu}} = \mathrm{diag}(-1, 1, 1, 1)$ and vectors has $A^{\widehat{\mu}} = A_{\widehat{\mu}}$ for all $\mu \in \{r, \theta, \phi\}$.

Ampere's law from classical Maxwell's equations can be written as (Jackson, 1998)

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \qquad (\mathbf{B} = \nabla \times \mathbf{A}), \tag{5}$$

where $\mathbf{B}$ is vector of magnetic field, $\mathbf{J}$ is vector of the current density and $\mathbf{A}$ is vector potential. Using Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the Ampere's law can be rewritten

$$\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}, \qquad (\nabla \times (\nabla \times \mathbf{A}) = \nabla \cdot (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}). \tag{6}$$

In SI units we are using vacuum permeability μ_0, in Gaussian units one must substitute μ_0 with $4\pi/c$.

Relativistic formulation of Maxwell's equations in flat spacetime is

$$\partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu} + \partial_\mu F_{\nu\alpha} = 0, \qquad \partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta. \tag{7}$$

where J^β is electric current four-vector. Electromagnetic tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{8}$$

where A^μ is electromagnetic four-vector. Assuming axial symmetry and absence of electric field, the only non-zero component of A^μ will be A^ϕ

$$A^\mu = (0, 0, 0, A^\phi), \quad A^\phi = A^\phi(r, \theta), \quad A_\phi = A^\phi\, r^2 \sin^2\theta = A^{\widehat{\phi}}\, r\sin\theta. \tag{9}$$

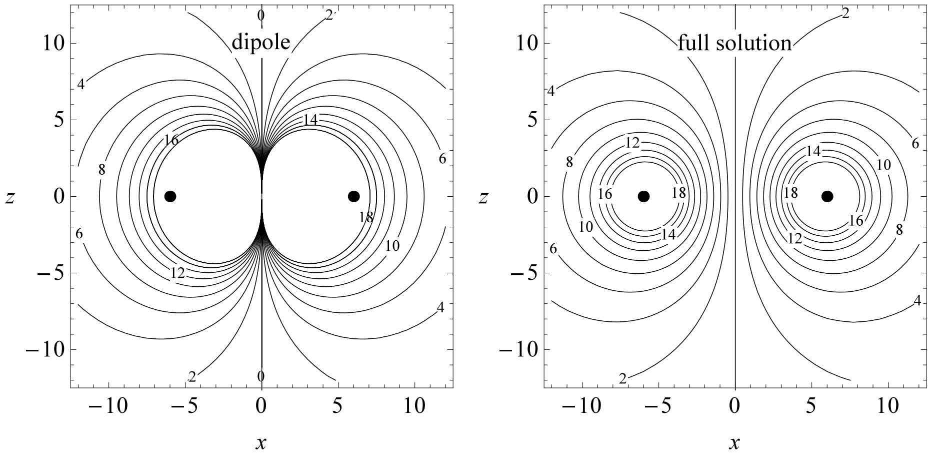

Figure 1. Shape of magnetic field lines around current loop with radius $a = 6$ in flat spacetime. Dipole approximation (left) and full analytic solution (right) are plotted.

The first of Maxwell's equations (7) is satisfied identically, while the second is giving the equation which we will be solving in this proceeding

$$r^2 \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} A_\phi\right) + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} A_\phi\right) = -\mu_0 J^\phi \, r^4 \sin^2\theta. \tag{10}$$

We will call this equation Ampere's law, but it can be also wield as special case of Grad—Shafranov equation well known from MHD (Meier, 2012).

Magnetic field three-vector $\mathbf{B} = (B^{\widehat{r}}, B^{\widehat{\theta}}, B^{\widehat{\phi}})$ can be related to four-vector A_ϕ

$$B^{\widehat{r}} = \frac{A_{\phi,\theta}}{r^2 \sin\theta}, \quad B^{\widehat{\theta}} = -\frac{A_{\phi,r}}{r\sin\theta}, \quad B^{\widehat{\theta}} = 0. \tag{11}$$

Circular current loop with radius a is located in x-y plane (equatorial plane $\theta = \pi/2$). Although the current is created by moving charge, the loop itself is considered to be neutral. The current loop is given by current density J^μ, but due to symmetry, only J^ϕ component will be non-zero

$$J^\phi(r,\theta) = \frac{I}{r^2}\,\delta(r-a)\,\delta(\theta-\pi/2), \qquad \int J^{\hat\phi}\,\mathrm{d}\hat r\,\mathrm{d}\hat\theta = \int J^\phi r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta = I. \tag{12}$$

where δ is Dirac delta function and the total current through r-θ plane (x-z plane) is normalized to I. Since the current density is given by delta functions $\delta(r-a)$ and $\delta(\theta-\pi)$, it is zero except for one point $r = a$ in equatorial plane. We will be looking for the solution of eq. (10) with the right side equal to zero, and the current existence will be important as boundary condition only.

3 SOLUTIONS OF MAXWELL'S EQUATIONS

In classical electromagnetism we could express the Ampere's law in integral form. The analytic solution for vector potential **A** can be found using

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\tilde{\mathbf{x}})}{|\mathbf{x} - \tilde{\mathbf{x}}|} \, \mathrm{d}^3 \tilde{x}. \tag{13}$$

The exact solution for circular current loop can be found using complete elliptic integral of the first $K(m)$ and second $E(m)$ kind (Jackson, 1998)

$$A^{\widehat{\phi}}(r,\theta) = \mu_0 I a \, \frac{(2-k^2)K(k^2) - 2E(k^2)}{\pi k^2 \sqrt{a^2 + 2ar\sin\theta + r^2}}, \qquad k^2 = \frac{4ar\sin\theta}{a^2 + r^2 + 2ar\sin\theta}. \tag{14}$$

Far away from the current loop $r >> a$ ($\mathbf{x} \approx \tilde{\mathbf{x}}$) simpler expression can be found

$$A^{\widehat{\phi}}(r,\theta) = \frac{\mu_0 I a^2 r \sin\theta}{4r^3}, \qquad \mathbf{A} = \frac{\mu_0}{4\pi r^2} \frac{\mathbf{m} \times \mathbf{r}}{r}, \tag{15}$$

where **m** current loop magnetic dipole moment $\mathbf{m} = \pi I a^2 \mathbf{z}$, **z** is unit base vector in z direction and $4\pi r^2$ is the surface of a sphere with radius r.

Assuming the current J^{ϕ} to be completely zero (no current loop), there exist another very simple solution to the Ampere's law (10)

$$A^{\widehat{\phi}}(r,\theta) = \frac{B}{2}\, r\sin\theta = \frac{B}{2}\, x, \qquad B^{\widehat{r}} = B\cos\theta, \quad B^{\widehat{\theta}} = -B\sin\theta. \tag{16}$$

This form of vector potential $A^{\widehat{\phi}}$ represent uniform magnetic field with strength B oriented perpendicularly to the equatorial plane (Wald, 1974)

Magnetic field **B** is fully specified by electromagnetic four-potential A^{μ}, see eq (11). We can provide the exact form of magnetic field **B**, but it is more elegant to work with electromagnetic potential A^{μ} instead. Shape of magnetic field **B** can be easily plotted using contour lines of electromagnetic potential

$$A^{\widehat{\phi}}(r,\theta) = \mathrm{const.} \tag{17}$$

For example for uniform magnetic field (16), equally spaced straight lines parallel to z axis can be obtained. For compassion the full analytic solution of vector potential (14) and its dipole approximation (15) are plotted in Fig. 1.

3.1 Multipole expansion

Using substitution $A_{\phi} = R(r) \cdot \Theta(\theta)$ for the electromagnetic four-potential A_{ϕ}, the Ampere's law (10) can be separated in two second order linear ordinary differential equations. The solution can be written as multipole expansion into spherical harmonics (Jackson, 1998)

$$A_{\phi} = \mu_0 J \sum_{n=0,2}^{\infty} \frac{\sqrt{\pi}\sin\theta \, P^1_{n+1}(0)\, Y^1_{n+1}(\theta,0)}{\sqrt{(n+1)(n+2)(2n+3)}} \cdot \begin{cases} r^{n+2}\, a^{-n-1} & \text{for} \quad r \in (0,a) \\ a^{n+2}\, r^{-n-1} & \text{for} \quad r \in (a,\infty) \end{cases} \tag{18}$$

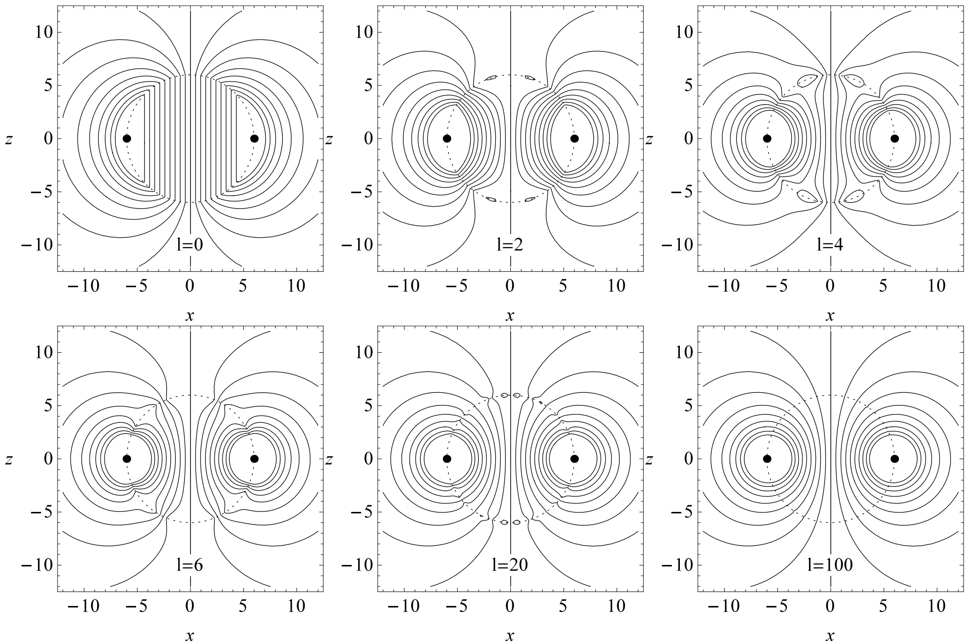

Figure 2. Current loop magnetic field shape for different orders of multipole expansion.

where $Y_n^1(\theta, 0)$ are Laplace's spherical harmonics (contains trigonometric functions) and $P_n^1(0)$ are associated Legendre polynomial (number coefficients). The same solution can be also expressed in different infinite series form (Petterson, 1974)

$$A_\phi = \frac{-\mu_0 J}{\sqrt{\pi}} \sum_{l=0,2}^{\infty} \mathrm{M}(l) \cdot \sin^2\theta \cdot \mathrm{C}_1^{\frac{3}{2}}(\cos\theta) \cdot \begin{cases} a^{-2l-3} r^{l+2} & \text{for} \quad r \in (0, a) \\ r^{-l-1} & \text{for} \quad r \in (a, \infty) \end{cases} \tag{19}$$

where $\mathrm{C}_1^{3/2}(\cos\theta)$ are Gegenbauer polynomials (special case of the Jacobi polynomials), $\Gamma(n)$ is Euler gamma function and the coefficients $\mathrm{M}(l)$ are given by

$$\mathrm{M}(l) = (-1)^{\frac{l+2}{2}} \frac{\left(l + \frac{3}{2}\right) \Gamma\left(\frac{1}{2} l + \frac{1}{2}\right)}{(l+2) \left(\frac{l}{2}\right)! \, (2l+3)} a^{l+2}. \tag{20}$$

Multipole expansion of A_ϕ consist of two parts: inner $r < a$ and outer $r > a$; only even terms will contribute to the sum while all odd terms are zero. While every each term of the infinite sum (18) is solution of (10) equation, the matching of inner and outer solution is not smooth for individual terms. Only if the total sum is taken into account, the discontinuity at sphere with radii $r = a$ will disappear. Magnetic fields for different terms of multipole expansions (18) are plotted in Fig. 2, where we clearly see how the discontinuity at $r = a$ radii disappear with increasing order of expansion and finally reaching the full analytic shape from Fig. 1.

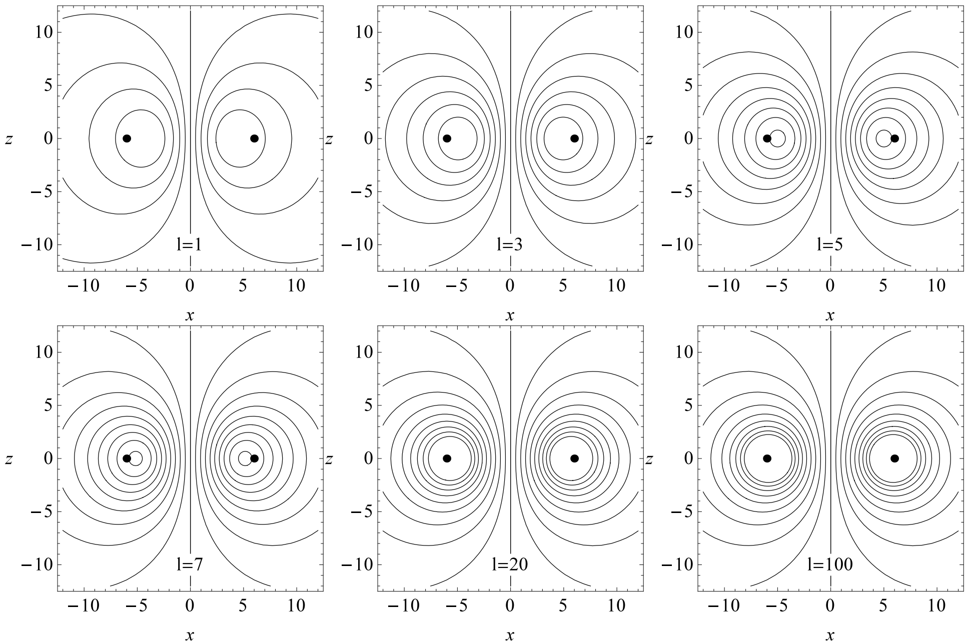

Figure 3. Current loop magnetic field shape for different orders of power series expansion.

Fist terms of the (18) and (19) series are

$$r \in (0,a) \quad A^{\widehat{\phi}} = \frac{\mu_0 J r \sin\theta}{4a} + \ldots \qquad r \in (a,\infty) \quad A^{\widehat{\phi}} = \frac{\mu_0 J a^2 \sin\theta}{4r^2} + \ldots \tag{21}$$

As we can see, the first term stands for uniform magnetic field ($B = \mu_0 J/2a$) in the case of inner solution $r < a$, while dipole field in the case of outer solution $r > a$.

3.2 Power series expansion

An alternative expansion was mentioned in (Jackson, 1998). Using powers series expansion, the solution to the (10) equation can be written as

$$A_{\phi(n)}(r,\theta) = \frac{\mu_0 J}{2} \sum_{l=1,2}^{n} \frac{\pi\left[(-1)^{n+1}+1\right](2n)!}{(n!)^2\,\Gamma\left(-\frac{n}{2}\right)^2\Gamma(n+2)} \cdot \frac{(ar\sin\theta)^{n+1}}{(a^2+r^2)^{n+\frac{1}{2}}} \tag{22}$$

where $\Gamma(n)$ is Euler gamma function and $n \to \infty$. All even terms are zero and only odd terms will contribute to the sum. Fist terms of the (22) expansions are

$$A^{\widehat{\phi}}(r,\theta) = \frac{\mu_0 J a^2 r \sin\theta}{4\,(a^2+r^2)^{3/2}} + \frac{15\mu_0 J a^4 r^3 \sin^3\theta}{32\,(a^2+r^2)^{7/2}} + \frac{315\mu_0 J a^6 r^5 \sin^5\theta}{256\,(a^2+r^2)^{11/2}} + \ldots \tag{23}$$

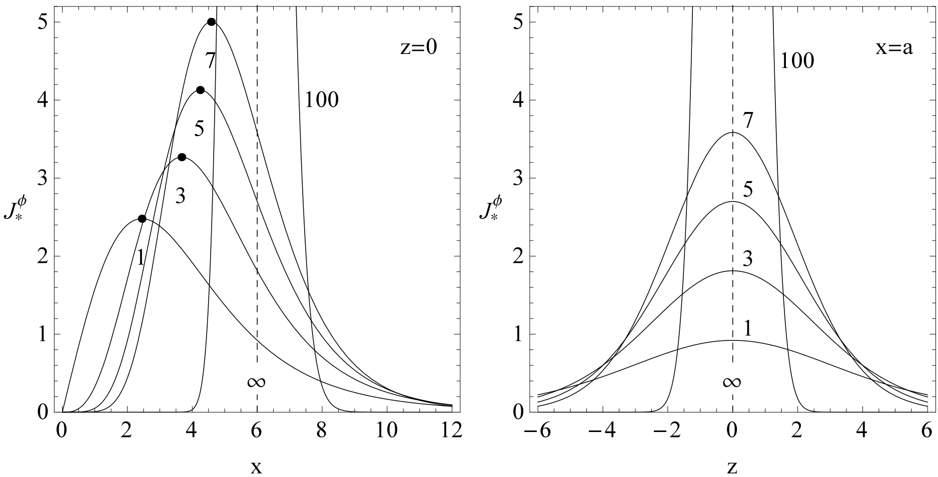

Figure 4. Different modes of current density $J^{\phi}_{*(n)}(r,\theta)$ for power series expansion (22).

While any individual term of (22) expansion are smooth function of r and θ coordinates, only the total sum A_ϕ is solution of (10) equation with current (12). Individual terms of (22) expansion are solutions of (12) equation with current J^ϕ not in thin steep delta function shape, but given by

$$J^{\widehat{\phi}}_{(n)}(r,\theta) = \frac{2a^3 I}{\sqrt{\pi}} \frac{\Gamma\left(n+\frac{5}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+3}{2}\right)} \frac{(ar\sin\theta)^n}{(a^2+r^2)^{n+\frac{5}{2}}}. \tag{24}$$

The current distribution $J^{\phi}_{*(n)}$ has been already summed up. The maximum of the current distribution function $J^{\phi}_{*(n)}(r,\theta)$, for individual term n of power expansion (22), is located in equatorial plane and below the actual current loop position a

$$r_{\max(n)} = a\sqrt{n}/\sqrt{n+5}, \quad \theta_{\max(n)} = \pi/2. \tag{25}$$

Also maximum of the electromagnetic four-potential $A^\phi(r,\theta)$ is shifted below the actual position of the current loop and its peak value is level down.

Magnetic field shape for different terms in power series expansion (22) is plotted in Fig. 3. As we can see, even the first term of power series expansion (22) is in very good agreement with the full analytic solution (14), if you are not close to position of the current loop. Contrary to the multipole expansion (18), the terms of power expansion are smooth and suitable for charged particle trajectory calculation.

4 CONCLUSIONS

The physics is the art of neglecting and hence in many physical models only the most relevant first terms of expansion are taken into account. Let assume, that the black hole magnetosphere has been generated by toroidal currents inside the accretion disk. With multipole expansion (18) we could use uniform magnetic field when we study charged particle motion below (inside) the current loop, while dipole magnetic field when studying particle moving above (outside) the loop.

Simple model of relativistic jet as a stream of charged particles escaping the inner parts of accretion disk along uniform magnetic filed lines has been studied in (Stuchlík and Kološ, 2016). More realistic black hole magnetosphere then simple uniform one should be considered, for example magnetosphere generated by toroidal current at radius a. But in multipole expansion the charged particle escaping from inner region of accretion disk (below the current loop) to infinity (above the current loop) will feel the discontinuity at sphere $r = a$. On the contrary, the power series expansion (22) is smooth everywhere and hence much more suitable for this charged particle jet model.

Multiple expansion of magnetic filed generated by current loop in black hole background has been already studied in literature, see summary in (Pejcha, 2014). This contribution has been done for flat spacetime only, focusing on power series expansion (22) and descriptive figures. In flat spacetime the full analytic solution si known and hence we can easily draw a comparisons of both multipole and power expansions with full analytic solution. We are now working on the magnetic field power series expansion, but in the Schwarzschild black hole spacetime.

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