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On the synchrotron radiation reaction in external magnetic field

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ABSTRACT
We study the dynamics of point electric charges undergoing radiation reaction force due to synchrotron radiation in the presence of external uniform magnetic field. The radiation reaction force cannot be neglected in many physical situations and its presence modifies the equations of motion significantly. The exact form of the equation of motion known as the Lorentz-Dirac equation contains higher order Schott term which leads to the appearance of the runaway solutions. We demonstrate effective computational ways to avoid such unphysical solutions and perform numerical integration of the dynamical equations. We show that in the ultrarelativistic case the Schott term is small and does not have considerable effect to the trajectory of a particle. We compare results with the covariant Landau-Lifshitz equation which is the first iteration of the Lorentz-Dirac equation. Even though the Landau-Lifshitz equation is thought to be approximative solution, we show that in realistic scenarios both approaches lead to identical results.

Keywords: radiation reaction – point charge – magnetic field

1 INTRODUCTION

Recent developments in accelerators and laser techniques with high intensities opened up many new phenomena which requires the inclusion of radiation reaction force into equations of motion of charged particles. Estimates confirm that in many physically relevant scenarios the radiation reaction force cannot be neglected. The study of the problem of the motion of electric charges with the effects of the radiation reaction force started more than century ago for nonrelativistic particles by Abraham and Föppl (1905), Lorentz (2003) and generalized by Dirac (1938) for relativistic case (for historical review, see Hammond (2010)). The exact form of equations plagues from the runaway solutions due to the presence of higher order term and may violate causality. The recent study of the radiation reaction in flat spacetime can be found in Gal'tsov and Spirin (2006) where the explicit derivation of the third-order Schott term was given along the lines of the Teitelboim idea to associate it with the Coulomb part of the electromagnetic field of the point charge. According to Teitelboim (1970) the Schott term arises from the derivative of the bound elec-

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tromagnetic momentum of the charge. The problem of the contribution of Schott term to the radiation reaction force still remains the source of discussion (see e.g. Poisson (2004)). In the present paper we will re-examine the relevance of this term too.

In order to avoid the non-physical solutions, Landau and Lifshitz (1975) proposed the iterated formula for the radiation reaction force. Their reduced order equation become first successful attempt to keep off the runaway solutions without violation of the principle of inertia (see discussions in Rohrlich (2001); Poisson (1999)).

In the present paper we are aimed to compare these two main approaches to the problem of the radiation reaction in the presence of external magnetic field. For doing so, we start from the general case and later solve the equations of motion numerically. For simplified scenario of the uniform magnetic field we find explicit trajectories of the charged particles together with the cooling times and related rates of the energy-momentum loss.

Generalizations of the radiation reaction to higher dimensions are given in Gal'tsov (2002); Gal'tsov and Spirin (2007). Radiation from hypothetical massless charges was studied in Gal'tsov (2015). The self force of point particles with mass, scalar and electric charges, has been reviewed in Poisson (2004). The electromagnetic radiation-reaction of extended classical charged particles has been studied in Cremaschini and Tessarotto (2011), its quantum approach is discussed in Cremaschini and Tessarotto (2015). Statistical treatment of the self-force is studied in Cremaschini and Tessarotto (2013). Nevertheless, despite the active interest in topic and broad bibliographical reviews, the successful attempts to integrate the equations of motion in realistic cases and presence of external electromagnetic fields is quite rare.

Throughout the paper we use the spacelike signature $(-, +, +, +)$, and the system of units with $c = 1$. However, for expressions having a physical relevance we use the constants explicitly. Greek indices are taken to run from 0 to 3.

2 SELF-FORCE OF A POINT CHARGE

Accelerated particles emit electromagnetic radiation leading to the radiation reaction force. In case of the circular motion the particle spirals down to the center of the motion due to the loss of energy and momentum. Generally, the charged particle equations of motion contain two forces

$$\frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} = f^{\mu}_{L} + f^{\mu}_{R}, \tag{1}$$

where $f^{\mu}_{L} = (q/m)F^{\mu\nu}u_{\nu}$ is the Lorentz force reflected by the external electromagnetic tensor $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ and $u^{\mu}(\tau) = dx^{\mu}/d\tau$ is a four-velocity of the particle. The last term is the radiation reaction force. In this section we introduce Lorentz-Dirac and Landau-Lifshitz equations for the self-force of point charge. The comparison of two approaches and integration of equations of motion are given in section 3.

2.1 Lorentz-Dirac equation

The radiation reaction force in the non-relativistic case is given by the expression $\frac{3q^2}{2m}\frac{d^2u^{\alpha}}{d\tau^2}$. In the relativistic case, any force vector has to satisfy the condition $f^{\mu}_{R}u_{\mu} = 0$. This implies that the correct covariant form of the expression for the radiation reaction force is

$$f_R^\mu = \frac{2q^2}{3m}\left(\frac{d^2u^\mu}{d\tau} + u^\mu u_\nu \frac{d^2u^\nu}{d\tau}\right). \tag{2}$$

This expression was found by Dirac and sometimes called as Lorentz-Abraham-Dirac or Lorentz-Dirac (LD) equation. The first term in the parentheses, also known as Schott term arises from the particle electromagnetic momentum. The second term in parentheses is the radiation recoil term which corresponds to the relativistic corrections of the radiation reaction force. The equation of motion of radiating particle is thus a third order differential equation in coordinates, rather than habitual second order. Equation (2) can be simplified reducing the order of the last term in the parentheses using the normalization condition. The four-velocity of the charged particle satisfies the following equations

$$u_\alpha u^\alpha = -1, \quad u_\alpha \dot{u}^\alpha = 0, \quad u_\alpha \ddot{u}^\alpha = -\dot{u}_\alpha \dot{u}^\alpha. \tag{3}$$

Substituting the last equality of (3) to the self-force (2) we get

$$f_R^\mu = \frac{2q^2}{3m}\left(\frac{d^2u^\mu}{d\tau} - u^\mu \frac{du_\nu}{d\tau}\frac{du^\nu}{d\tau}\right). \tag{4}$$

Since the Schott term is still of the third order in coordinates the expression (2) leads to the well known problem of the classical electrodynamics - the existence of so called runaway solutions which violate causality. Unphysical solutions appear due to the possible presence of the pre-acceleration in the absence of external forces. Even though the unphysical solutions can be removed by properly chosen values of initial conditions, the exact form of equations of motion (1) and (2) is inconvenient to solve due to the presence of higher order term which exponentially increases the computational error in practical calculations.

2.1.1 *Synchrotron radiation without self-force*

The Lorentz-Dirac equation can lead to the following interesting consequence. Introducing the four acceleration as $a^\mu = du^\mu/d\tau$ for a particle moving in Minkowski space one can write the rate of radiated four-momentum from particle and the radiation reaction force acting on a particle in the following form

$$\frac{dP^\mu}{d\tau} = \frac{2q^2}{3} a^\alpha a_\alpha u^\mu. \tag{5}$$

$$f_R^\mu = \frac{2q^2}{3m}\left(\frac{da^\mu}{d\tau} + u^\mu u_\nu \frac{da^\nu}{d\tau}\right). \tag{6}$$

Equation (5) also allows to find the rate of energy and angular momentum loses, alternatively from the way derived in the later sections, where the cooling rates are found directly from the equations of motion. One can notice that the radiation reaction force vanishes in case of uniform acceleration while radiated power remains non-zero. This leads to the energy loss with conserved angular momentum. Thus, the radiating charged particle may or may not feel the radiation reaction force. However, this case corresponds to the full form of Eq.(6) which as pointed above leads to the appearance of the runaway solutions and, moreover, corresponds to the improper choice of initial conditions. In case when the initial acceleration of a particle is chosen to be zero, which as we will see below corresponds to the only reasonable choice of initial conditions, then the problem described in this subsection disappears. Equivalently, when the reduction of order procedure is applied to the equations of motion, then any radiating accelerated particle with uniform or non-uniform acceleration will radiate the synchrotron radiation and undergo the radiation reaction force.

2.2 Landau-Lifshitz equation

In order to exclude the unphysical solutions and find equations of motion in usual second order form, it was proposed by Landau and Lifshitz to rewrite the radiation reaction force in terms of the external force and the four-velocity performing the iteration of the Lorentz-Dirac equation. Substituting in (2) instead of the higher order terms the derivatives of the Lorentz force, we get the following equation of motion for the charged particle

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = f_L^\mu + \frac{2q^2}{3m}\left(\delta^\mu_\alpha + u^\mu u_\alpha\right)\frac{df_L^\alpha}{d\tau}. \tag{7}$$

This equation does not contain runaway solutions and has been proved to be useful in e.g. Landau and Lifshitz (1975), Poisson (1999). The important consequence of the equation above that it is of a second order, does not violate the principle of inertia and in the absence of the Lorentz force, the radiation reaction force vanishes as well Rohrlich (2001). The self-contained derivation of the equation (7) in terms of retarded potentials is given in Poisson et al. (2011). The equation (7) has more general property and can be applied for the cases with any external force acting on a charged particle instead of the Lorentz force. In case when $f_L^\mu = \frac{q}{m}F^{\mu\nu}u_\nu$, the radiation reaction force can be rewritten in the form

$$f_R^\mu = k\tilde{q}\left(F^{\mu\nu}{}_{,\alpha}u^\alpha u_\nu + \tilde{q}(F^{\mu\nu}F_{\nu\rho} - F^{\beta\alpha}F_{\beta\rho}u_\alpha u^\mu)u^\rho\right), \tag{8}$$

where $\tilde{q} = q/m$ is a specific charge of the particle, $k = (2/3)\,\tilde{q}\,q$, and the comma in the first term denotes the partial derivative with respect to the coordinate x^α. The characteristics of the motion of radiating charged particle undergoing a radiation reaction force (8) in particular representative case is studied in the following section.

3 INTEGRATION OF EQUATIONS AND RELATED TRAJECTORIES

One can basically find the trajectory of the radiating charged particle in two ways, either solving the third order equations (4) or using the iterated formula (8). Below we solve both equations numerically and compare results.

Let us consider a simple case of the motion of charged particle in flat spacetime filled with a homogeneous magnetic field aligned along z-axis, $\boldsymbol{B} = (0, 0, B)$. The tensor of electromagnetic field $F_{\alpha\beta}$ in this case will have only two nonzero components $F_{xy} = -F_{yx} = B$. Since the magnetic field B enters to the equations of motion together with the specific charge of charged particle one can introduce the new parameter in the form $\mathcal{B} = qB/(2m)$.

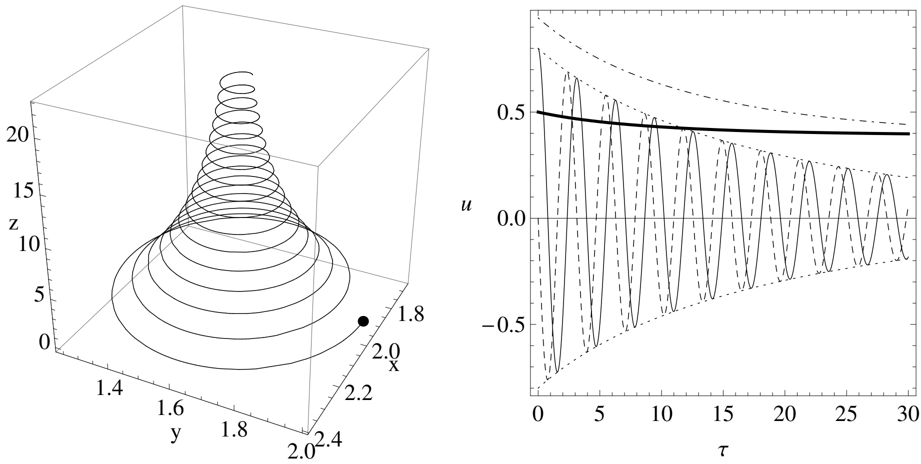

Figure 1. Motion of radiating charged particle in flat spacetime. Left figure represents an example of 3D trajectory of charged particle. Middle plot represents the evolution of different components of 3-velocity in proper time τ: the damping harmonic oscillations correspond to u^x (solid thin) and u^y (dashed) components of velocity, tangential to them (dotted) is a plane velocity of a particle $v_\perp$ orthogonal to z axis, vertical u^z component of velocity is shown by solid thick line and dot-dashed curve of the middle plot shows the evolution of 3-velocity ($\sqrt{u_x^2+u_y^2+u_z^2}$). Right figure shows the change of the energy, angular momentum and gyroradius of charged particle in time t with respect to observer at rest. corresponding to the trajectory illustrated in Fig.1. Energy decreases up to the nonzero irredusible energy level, while angular momentum and radius of gyration asymptotically tend to zero. For illustrative purposes we have set the radiation parameter k to the value much larger than in realistic situations.

Below, we define the proper velocity of a particle as $u^\alpha = dx^\alpha/d\tau$ and coordinate velocity (velocity measured by observer at rest) by $v^\alpha = dx^\alpha/dt$.

3.1 Integration of Lorentz-Dirac equations

The equations of motion (1) and (4) can be written explicitly in the form

$$\frac{du^x}{d\tau} = 2\mathcal{B}u^y - k\,a^2u^x + k\frac{d^2u^x}{d\tau^2}, \tag{9}$$

$$\frac{du^y}{d\tau} = -2\mathcal{B}u^x - k\,a^2u^y + k\frac{d^2u^y}{d\tau^2}, \tag{10}$$

$$\frac{du^z}{d\tau} = -k\,a^2u^z + k\frac{d^2u^z}{d\tau^2}, \tag{11}$$

$$\frac{du^t}{d\tau} = -k\,a^2u^t + k\frac{d^2u^t}{d\tau^2},. \tag{12}$$

where we used the notation

$$a^2 \equiv a_\alpha a^\alpha = -\left(\frac{du^t}{d\tau}\right)^2 + \left(\frac{du^x}{d\tau}\right)^2 + \left(\frac{du^y}{d\tau}\right)^2 + \left(\frac{du^z}{d\tau}\right)^2 . \tag{13}$$

In order to eliminate the unphysical solutions of the equations (9) - (12) one needs to choose the initial conditions of a charged particle properly. As for any third order ordinary differential equations in Minkowski spacetime, one needs to set the values of 9 constants - arbitrary independent components of initial position, velocity and the acceleration of the charged particle. The three other constants can be identified by normalization condition (3). The values of initial conditions which satisfy the physical solutions of equations of motion are those in which all components of initial acceleration are vanishing. However, the direct integration of higher order equations leads to the exponential increase of the computational error in very short times. We have found that the problem of the time dispersion error can be greatly reduced by integrating equations of motion backward in time. Similar method of solving higher order equations has been proposed in the past by Huschilt and Baylis (1976). The result of integration of equations of motion are shown in Fig. 1. It is interesting to note that the same result has been obtained using the Landau-Lifshitz approach, which is discussed below. We also leave the discussion of Fig. 1 to the following subsection. Proposed by Landau and Lifshitz (1975) as an approximative solution to the third order LD equations, it was more recently argued by Rohrlich (2001), that the reduced form of the equations of motion are exact, rather than approximative. In our calculations, corresponding to Fig.(1) the difference between two methods appears to be negligible.

3.2 Integration of Landau-Lifshitz equations

For the motion in a uniform magnetic field, the reduced-order equations of motion (7) of radiating charged particle can be written explicitly in the form

$$\frac{du^x}{d\tau} = 2\mathcal{B}u^y - 4k\mathcal{B}^2\left(1 + u_\perp^2\right)u^x, \tag{14}$$

$$\frac{du^y}{d\tau} = -2\mathcal{B}u^x - 4k\mathcal{B}^2\left(1 + u_\perp^2\right)u^y, \tag{15}$$

$$\frac{du^z}{d\tau} = -4k\mathcal{B}^2 u_\perp^2 u^z, \tag{16}$$

$$\frac{du^t}{d\tau} = -4k\mathcal{B}^2 u_\perp^2 u^t, . \tag{17}$$

Here $u_\perp^2 = (u^x)^2 + (u^y)^2$ is a square of plane velocity of a particle orthogonal to the magnetic field and z axis, $k = (2/3)\tilde{q}q$ is the parameter responsible for the radiation reaction. The representative view of corresponding trajectories is shown in Fig.1, which coincides with the results of integrations of the Lorentz-Dirac equations (see previous subsection). The particle starts its motion with the initial plane velocity $u_{\perp 0} = 0.8c$ and the vertical velocity $u_0^z = 0.5c$. Due to combined effects of the Lorentz force and the radiation reaction force, the charged particle spirals down to the center. It occurs due to the loss of energy and angular momentum of a particle and correspondingly, the plane velocity of a particle decreases by

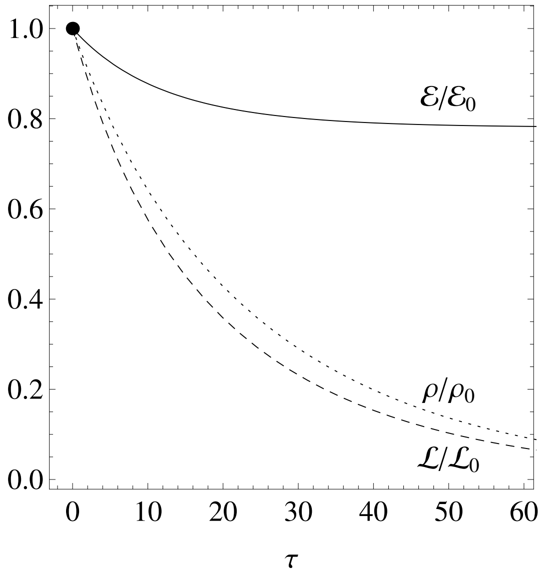

Figure 2. Figure shows the evolution of the energy, angular momentum and gyroradius corresponding to the trajectory illustrated in Fig.1. Energy decreases up to the nonzero irredusible energy level, while angular momentum and radius of gyration asymptotically tend to zero.

the law given as the solution of equations (14) and (15) or equivalently as solutions of (9) and (10). However, if one looks to the motion along z axis shown in Fig.1 (solid thick in the right plot), one can notice that the proper velocity of a charged particle $u^\alpha = dx^\alpha/d\tau$ undergoes small nonzero deceleration in the direction parallel to the magnetic field which coincides with z axis, while there are no external forces in z direction. This contradiction with the common sense occurs due to the reason that the system of radiating particle is non-conservative. The problem, however, vanishes if we measure the velocity of a radiating charged particle with respect to the static observer $v^\alpha = dx^\alpha/dt \equiv u^z/u^t = const$, where the motion appears to be uniform in the direction orthogonal to the action of the Lorentz force.

3.3 Energy and momentum loss

The rate of the energy loss of a particle can be easily evaluated from Eq.(12) or (17). However, one needs to note that it is more convenient to use the equation (17) as it is ordinary second order differential equation. Modifying it for a static observer we get

$$\frac{dE}{dt} = -kB^2\tilde{q}^2 u_\perp(t)^2. \tag{18}$$

Integrating this equation one can obtain the energy of a particle in a given moment of time. The representative plot of the evolution of the energy in time is given in Fig.2. Thus the energy loss will be given only by the change of the plane velocity $u_\perp$ in time, while the linear component of the velocity of a particle in z-direction will be conserved. This implies that the part of kinetic energy of a charged particle related to the linear motion along z axis is conserved as well. Writing the normalization condition $u^\alpha u_\alpha = -1$ explicitly and reminding that the particle specific energy is $\mathcal{E} = E/m = -u_t$, we get the formal decomposition of the total energy of the particle in the form

$$\mathcal{E}^2 = 1 + u_\perp^2 + u_z^2 \equiv 1 + \mathcal{E}_\perp^2 + \mathcal{E}_z^2, \tag{19}$$

where $u_\perp^2 = u_x^2 + u_y^2$. The final state of the particle is defined by the first and the last terms of the right hand side of Eq.(19). This implies that there exists an irreducible energy of radiating charged particle which corresponds to the final state of a particle having the following simple form

$$\mathcal{E}_{\rm irr} = \left(1 - (v_0^z)^2\right)^{-\frac{1}{2}}, \tag{20}$$

where v_0^z is the vertical velocity of the particle along the z-axis measured by static observer which is constant during the radiation process.

In order to find the rate of the angular momentum loss one can concentrate attention only on the motion in a fixed plane by taking $u^z = 0$. Thus, we have $u_\perp$ component of 3-velocity only. Angular momentum, in general, for the motion in plane is defined by formula $L = m\rho^2 d\phi/d\tau + qA_\phi$, where ρ is gyroradius of the particle trajectory. Reminding that $d\phi/d\tau = u_\perp\gamma/\rho$ and $\rho = u_\perp/\omega_L$, where $\omega_L = qB/m \equiv 2\mathcal{B}$ is the Larmor frequency, one can write the angular momentum of radiating charged particle in the given moment of time in the form

$$L = \frac{m u_\perp(\tau)^2}{4\mathcal{B}} \left[2\gamma(\tau) + 1\right], \quad \gamma = (1 - u_\perp^2)^{-\frac{1}{2}}. \tag{21}$$

Solving first two equations of motion (14) and (15) and substituting into Eq.(21) we get the evolution of angular momentum in time, which is represented in Fig.2. Unlike the energy of a particle, the angular momentum asymptotically tends to zero for large τ. This occurs due to the reason that the gyroradius ρ of a charged particle tends to zero as well, while L is proportional to ρ. The relative evolutions of the energy, angular momentum and gyroradius are illustrated in Fig.2.

One can find the ratio between the angular momentum loss $\dot{L} = dL/dt$ and energy loss $\dot{E} = dE/dt$ as

$$\frac{\dot{L}}{\dot{E}} = r^2\Omega \frac{u_\perp^2 + 1}{u_\perp^2}. \tag{22}$$

where Ω is an angular velocity of the charged particle measured by observer at rest. In Cartesian coordinates Ω takes the form

$$\Omega = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}, \tag{23}$$

where dots denote the derivative with respect to the coordinate time t. One can see from Eq.(22) that in the processes connected with the synchrotron radiation the angular momentum loss of a radiating particle cannot be neglected.

B (Gauss)	τ_e (s)	τ_p (s)
10^{12}	10^{-16}	10^{-6}
10^{8}	10^{-8}	10^{2}
10^{4}	1	10^{10}
1	10^{8}	10^{1}
10^{-4}	10^{16}	10^{26}

Table 1. Typical decay times of ultrarelativistic electrons τ_e and protons τ_p for different values of magnetic field B.

3.4 Cooling time

One can find the cooling time of a radiating charged particle in the following way. Since the velocity in the magnetic field direction is constant one can consider the motion of a particle in plane only by taking $u^z = 0$. This implies, that according to condition $u_\alpha u^\alpha = -1$ we get $u_\perp^2 = (u^t)^2 - 1$. Thus, the equation (17) can be rewritten in the form

$$\frac{d\mathcal{E}}{d\tau} = -\mathcal{K}\left(\mathcal{E}^3 - \mathcal{E}\right), \quad \mathcal{K} = 4k\mathcal{B}^2 \equiv \frac{2q^4B^2}{3m^3c^5}. \tag{24}$$

Integrating above equation we get the energy of a particle in a given moment of time

$$\mathcal{E}(\mathcal{T}) = \frac{\mathcal{E}_0 e^{\mathcal{K}\mathcal{T}}}{\sqrt{1 + \mathcal{E}_0^2\left(e^{2\mathcal{K}\mathcal{T}} - 1\right)}}, \tag{25}$$

where the integration constant $\mathcal{E}_0$ is the initial energy of the particle and $\mathcal{E}(\mathcal{T})$ is the energy in a given moment of time, which is an exponentially decreasing function of time $\mathcal{T}$. Asymptotically in time the energy tends to the particle rest energy, being equal to 1. Thus, from (25) we find the cooling time during which the energy will be lowered from $\mathcal{E}_0$ to $\mathcal{E}_f$ due to radiation in the following form

$$\mathcal{T} = \frac{1}{2\mathcal{K}} \ln \frac{\mathcal{E}_f^2(\mathcal{E}_0^2 - 1)}{\mathcal{E}_0^2(\mathcal{E}_f^2 - 1)}. \tag{26}$$

For example, the time required to decay the half of the initial energy of the particle ($\mathcal{E}_f = \mathcal{E}_0/2$) is

$$\mathcal{T}_{1/2} = \frac{1}{2\mathcal{K}} \ln \frac{\mathcal{E}_0^2 - 1}{\mathcal{E}_0^2 - 4}. \tag{27}$$

One can estimate the timescale of the decay of charged particle energy as follows. Assuming the ultrarelativistic particle with initial and final energies: $\mathcal{E}_0 \gg 1$ and $\mathcal{E} \approx 1$, the logarithm in (26) can be approximately taken to be unity. Then, the timescale of cooling of the charge is simply $\mathcal{T} \approx 1/\mathcal{K}$. Typical orders of magnitude of the oscillation decay time of an electron and proton are shown in Tab.1. The radiation reaction and related cooling time of charges can have a significant effect in plasma, when the decay time of charged particle becomes smaller than the average time between collisions of particles in plasma. For typical hydrogen plasma with the temperature $T = 10^8$K and concentration $n = 10^{14}\mathrm{cm}^{-3}$ the timescales of electron-electron, electron-ion and ion-ion collisions take the values

$$t_{ee} \approx 6.4 \times 10^{-4}\mathrm{s}, \quad t_{ei} \approx 4.5 \times 10^{-4}\mathrm{s}, \quad t_{ii} \approx 4.1 \times 10^{-2}\mathrm{s}. \tag{28}$$

Thus, comparing (28) with Tab.1, one can conclude that in many physically relevant scenarios the radiation-reaction force cannot be neglected.

4 CONCLUSIONS

We have studied the problem of radiation reaction of charged particle using two main approaches, given by the Lorentz-Dirac and Landau-Lifshitz equations. Comparing results in case of the uniform magnetic field we have found that the difference between two approaches is negligibly small. In order to solve the Lorentz-Dirac equations which are of the third order in coordinates we have performed the integration of dynamical equations backward in time. We have found that the physical solutions of Lorentz-Dirac equations are those which correspond to the initial conditions with zero initial acceleration or equivalently those where the acceleration is zero in an arbitrary given moment of time. We have also found that in ultrarelativistic case the contribution of Schott term is considerably small. Together with the numerical solutions of the dynamical equations of charged particle we have obtained analytical expressions for the energy and angular momentum loss of the particle and the corresponding cooling time. The solution of Landau-Lifshitz equations is straightforward and more convenient since the equations contain the physical solutions only and do not violate causality. Small difference between Lorentz-Dirac and Landau-Lifshitz approaches imply that in the most physically relevant situations one can use the Landau-Lifshitz equation instead of Lorentz-Dirac one. It was pointed out by Spohn (2000) that using the Landau-Lifshitz equation is identical to imposing Dirac's asymptotic condition $\lim_{\tau\to\infty} \dot{u}^{\mu} = 0$ to the Lorentz-Dirac equation. This implies that the Landau-Lifshitz equation is exact, rather than approximative. One can conclude that this statement is correct at least in the case when the Schott term is small, which is exactly the case we have shown.

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