Gravitational field energy of rotating brany black hole

Bobur Turimov,[1,a] Javlon Rayimbaev[1,b]
and Azam Rakhmatov[2,c]

[1] Ulugh Beg Astronomical Institute, Astronomicheskaya 33, Tashkent 100052, Uzbekistan
[2] National University of Uzbekistan, Tashkent,100174, Uzbekistan
[a] bturimov@astrin.uz
[b] javlon@astrin.uz

ABSTRACT
In the present paper we investigate the gravitational field energy associated with the rotating brany black hole by using Tolman's and Landau-Lifshitz's approaches. It is shown that both approaches give exactly the same results and the total gravitational energy is found to be shared by its interior as well as exterior spacetime of the black hole. Switching off the brane parameter (i.e., $Q^* = 0$) one finds that there is no energy shared by the exterior of the rotating brany black hole. Thus the gravitational field seems to have a remarkable difference in comparison the electromagnetic field as the energy in the latter case is shared by the interior as well as exterior of the system producing the electromagnetic field .

Keywords: Brany black hole – Energy associated – Gravitational energy

1 INTRODUCTION

One of the important properties of black holes is their energy associated with exterior spacetime. Using Tolman's definition one can find an expression of the energy associated with a general non-static spherically symmetric spacetime, which is enable to get the energy of radiating Schwarszchild and brany black hole spacetimes. The calculation demonstrates that the energy of a nonlocal system is shared by its exterior as well as interior but for a purely gravitating system the entire energy is confined to its interior.

In the papers Lynden-Bell and Katz (1985); Dadhich and Chellathurai (1986) it has been studied a coordinate-independent definition of the gravitational field energy of the static spherically symmetric spacetime and then it has been shown that the entire gravitational field energy of a Schwarzschild and Reissner-Nordström black hole remains outside the hole. Although there is no unique way of defining energy in curved spacetime, that is why in order to compute the energy one have to work on Cartesian spacetimes. In the paper Virbhadra (1990) it has been evaluated the energy associated with the Kerr-Newman

978-80-7510-257-7

spacetime (charged Kerr spacetime) correction; though and has found that the charge of the black hole strongly depends on the energy associated.

In this paper we are interested in study of the gravitational field energy of the black hole in one of the alternate theories of gravity, in which braneworld model proposed by Randall and Sundrum (1999). In this model the matter is confined to a three dimensional braneworld space, embedded in a larger space so-called bulk in which only gravitation interaction can propagate. The static and spherically symmetric exterior solution of the brane world models has been obtained in Dadhich et al. (2000) in astrophysical scale, which exactly coincides with the Reissner-Nordström solution with the only difference that the brane parameter Q^* stands instead of the square of the electric charge Q^2.

In the present work we will show the effect of the brane parameter in the energy associated with exterior spacetime of the rotating brany black hole and its gravitational energy. First we will show cal calculation of the energy associated with with exterior spacetime the rotating brany black hole in two different ways, such that by using Tolman's and Landau-Lifshitz's approach, then we will compute the gravitational field energy by the exterior spacetime of the brany black hole. Throughout this paper we have used signatures $(-,+,+,+)$ for the spacetime and geometrized unit system $G = c = 1$. Latin indexes run from $1,2,3$ and Greek ones from 0 to 3.

2 THE ROTATING BRANY BLACK HOLE

In this section we will briefly mention about the spacetime of the rotating brany balck hole and give a definition of the energy associated by the black hole through its external spacetime. In the spherical coordinate (t, r, θ, ϕ) the spacetime geometry of the rotating brany black hole can be given by the following line element

$$\begin{aligned} ds^2 =& \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{1}{\Sigma}\left[(\Sigma + a^2\sin^2\theta)^2 - \Delta a^2\sin^2\theta\right]d\phi^2 \\ &- \frac{2}{\Sigma}\left[\Sigma + a^2\sin^2\theta - \Delta\right]d\phi dt - \frac{1}{\Sigma}\left[\Delta - a^2\sin^2\theta\right]dt^2 , \end{aligned} \tag{1}$$

where $\Delta = r^2 - 2Mr + a^2 + Q^*$, $\Sigma = r^2 + a^2\cos^2\theta$, and M is the total mass and a is the specific angular momentum of the brany black hole or the spin parameter. The quantity Q^* is the brane charge parameter which is negatively defined $Q^* \leq 0$. One can easily see that the metric in the expression (1) is very similar with Kerr-Newman solution, difference is the square of the electric charge is replaced by brane charge parameter. The Kerr solution can be obtained in the case when the brane charge parameter vanishes $Q^* = 0$.

Since the brane charge parameter takes negative value the radius of the horizon can be found as

$$r_+ = M\left(1 + \sqrt{1 - \frac{a^2}{M^2} + \frac{|Q^*|}{M^2}}\right) \tag{2}$$

from equation (2) one can easily see that spin parameter a of the brany black hole can be greater than its mass as $0 \leq a \leq \sqrt{M^2 + |Q^*|}$. In the paper Turimov et al. (2017) it is found

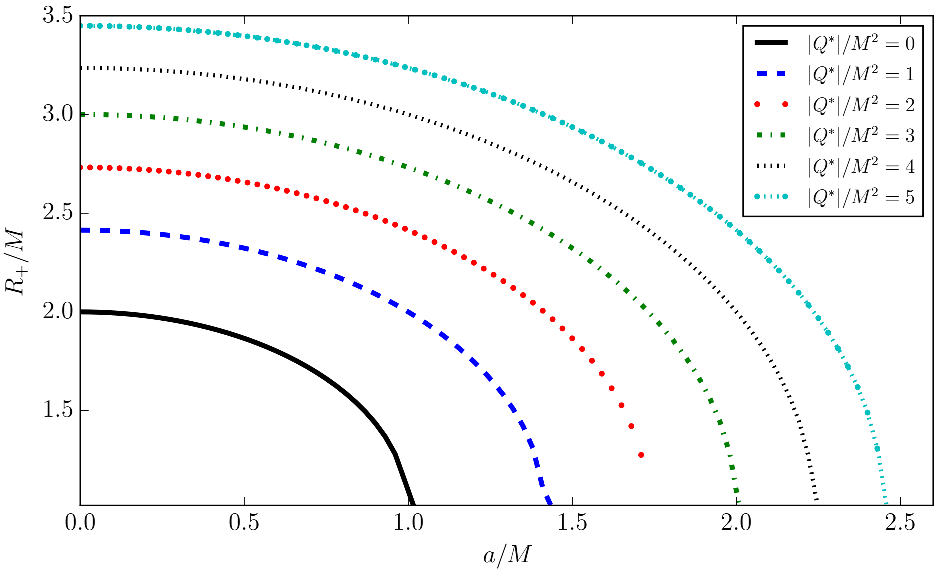

Figure 1. Dependence of the radius of the horizon on the spin parameter for different values of the brane charge parameter.

the upper limit for the value of the brane charge parameter is approximately $|Q^*| \lesssim 8M^2$, that means the spin of the black hole $0 \leq a \leq 3M$. In Fig.1 dependence of the radius of the horizon on the spin parameter for the different values of the brane parameter is shown. One can easily see that the spin of the black hole can take the value up $a \lesssim 3M$. While Figure 2 draws dependence of the radius of the horizon on the brane parameter for the different values of the spin parameter is shown. One can easily see that the size of the brany black hole larger than that in Kerr black hole.

In the present research our aim is investigating the gravitational field energy of the brany black hole, such a work performed by Lynden-Bell and Katz, that main idea is the total energy $E_0 = M$ (in the unit $c = 1$) of the the black hole is sum of the matter energy E_m and the gravitational field energy E_f that can be expressed as Lynden-Bell and Katz (1985); Dadhich and Chellathurai (1986)

$$E_0 = M = E_m + E_f \tag{3}$$

In order to calculate the gravitational energy one have to compute the energy associated with exterior spacetime of the black hole that can be evaluated in the following form

$$E_m = \int d^3x \sqrt{-g}\, T^{00} \tag{4}$$

where $T^{\mu\nu}$ is the energy-momentum tensor that could be found from Einstein field equations for given metric.

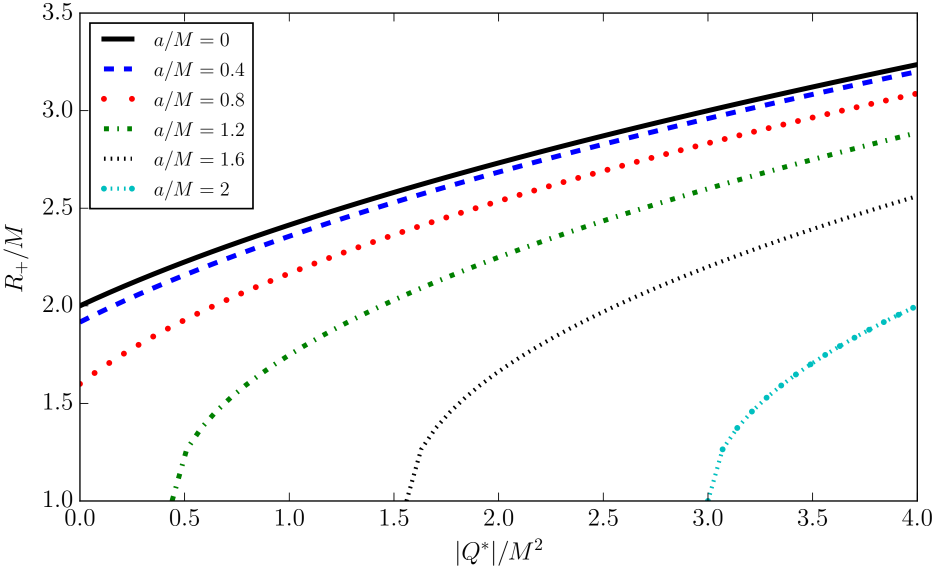

Figure 2. Dependence of the radius of the horizon on the brane charge parameter for different values of the spin parameter.

In the present work we will show the calculation of the gravitational field energy in two different ways: (i) Tolman's approach and (ii) Landau and Lifshitz's approach. In both approaches it is required that to get correct value of the energy the spacetime around the black hole should be considered in Cartesian coordinates (t, x, y, z), that the metric around rotating brany black hole (1) can expressed as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \frac{(2Mr_0 - Q^*)r_0^2}{r_0^4 + a^2z^2} \times \left[dt + \frac{z}{r_0}dz + \frac{xr_0 - ya}{r_0^2 + a^2}dx + \frac{yr_0 + xa}{r_0^2 + a^2}dy\right]^2 , \tag{5}$$

where

$$r_0^4 - r_0^2(x^2 + y^2 + z^2 - a^2) - a^2z^2 = 0 . \tag{6}$$

3 TOLMAN'S APPROACH

In this section we will show the calculation of the energy associated by the brany black hole in Tolman's approach and the definition of the energy E_m associated with arbitrary space-time is written in the form Tolman (1930)

$$E_m = \frac{1}{8\pi}\int \partial_\nu \left(U_0^{0\nu}\right) d^3x\,, \tag{7}$$

here the pseudotensor $U_\mu^{\alpha\beta}$ is defined as

$$U_\mu^{\alpha\beta} = \sqrt{-g}\left(-g^{\nu\alpha}V_{\beta\nu}^{\beta} + \frac{1}{2}g_\mu^\alpha\, g^{\nu\sigma}V_{\nu\sigma}^{\beta}\right) \tag{8}$$

with

$$V_{\alpha\beta}^{\mu} = -\Gamma_{\alpha\beta}^{\mu} + \frac{1}{2}g_\alpha^\mu\,\Gamma_{\nu\beta}^{\nu} + \frac{1}{2}g_\beta^\mu\,\Gamma_{\nu\alpha}^{\nu} \tag{9}$$

where $\Gamma_{\alpha\beta}^{\mu}$ Cristoffel symbol. Before we mentioned that the energy associated in Tolman's approach can be evaluated in Cartesian coordinates only, obviously without any approximation the calculation will be complicated. That is why we assume that spin parameter quantitatively very small for most physical situations and to be convenience we consider terms containing powers of a up to a^3 only. Since we are interested in calculating the energy associated with the exterior of the rotating brany black hole, the mass of the black hole M and the brane charge parameter Q^* are constants. Now we evaluate all the required components of affine connection, neglecting terms containing powers of a greater than 3.

Nonzero components of affine connection U_0^{0i} (i=1,3) are

$$U_0^{01} = \frac{2Mx}{r^3} - \frac{Q^*x}{r^4} - \frac{aMy}{r^4} + \frac{2a^2Q^*x}{r^6}\left(\frac{3z^2}{r^2} - 1\right) - \frac{2a^2Mx}{r^5}\left(\frac{5z^2}{r^2} - 1\right) - \frac{a^3My}{r^6}\left(\frac{3z^2}{r^2} - 1\right), \tag{10a}$$

$$U_0^{02} = \frac{2My}{r^3} - \frac{Q^*y}{r^4} + \frac{aMx}{r^4} + \frac{2a^2Q^*y}{r^6}\left(\frac{3z^2}{r^2} - 1\right) - \frac{2a^2My}{r^5}\left(\frac{5z^2}{r^2} - 1\right) - \frac{a^3Mx}{r^6}\left(\frac{3z^2}{r^2} - 1\right), \tag{10b}$$

$$U_0^{03} = \frac{2Mz}{r^3} - \frac{Q^*z}{r^4} + \frac{aMx}{r^4} + \frac{2a^2Q^*z}{r^6}\left(\frac{3z^2}{r^2} - 1\right) - \frac{2a^2Mz}{r^5}\left(\frac{5z^2}{r^2} - 1\right), \tag{10c}$$

Further inserting equation (10) into (7) and then transforming it in spherical coordinates one can get following expression

$$E_m = \frac{1}{8\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_{r_+}^{\infty} dr\left[\frac{2a^2Q^*}{r^4}(3\cos^2\theta - 1) + \frac{Q^*}{r^2}\right] \tag{11}$$

After calculating the integral in the equation (11) one can find the energy associated with exterior rotating brany black hole in Tolman's approach

$$E_m = -\frac{|Q^*|}{2r_+}\left(1 + \frac{2a^2}{3r_+^2}\right), \tag{12}$$

where r_+ is the radius of the horizon of the rotating brany black hole which is given in the equation (2). From the expression (12) one can see that the energy associated with exterior spacetime rotating brany black hole should be also negative.

4 LANDAU-LIFSHITZ'S APPROACH

Another way of computing the energy associated with exterior spacetime of the black hole has been performed by Landau-Lifshitz, which is much more easy to work out in comparison with that in Tolman's way of calculation. In this approach Einstein's field equations can be written in the form Landau and Lifshitz (2004)

$$\partial_\alpha \partial_\beta \mathcal{H}^{\mu\alpha\nu\beta} = 16\pi \mathcal{T}^{\mu\nu} \quad (13)$$

where $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor of matter and $\mathcal{H}_{\mu\nu\alpha\beta}$ is the Riemann tensor which is defined by in the following form

$$\mathcal{H}^{\mu\alpha\nu\beta} = \mathbf{g}^{\mu\nu}\mathbf{g}^{\alpha\beta} - \mathbf{g}^{\mu\beta}\mathbf{g}^{\alpha\nu} \quad (14)$$

where $\mathbf{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$. The energy associated by the black hole is calculated by

$$E_m = \int \mathcal{T}^{00} d^3x \quad (15)$$

These integral also have the meaning of the energy only if it is evaluated in Cartesian coordinates. Considering the terms containing powers of the spin parameter up to a^3 and evaluating nonzero components of the tensor $\mathcal{H}^{0i0j}$ one can rewrite equation (15) in the following form

$$E_m = \frac{1}{8\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_{r_+}^{\infty} dr \left[\frac{2a^2 Q^*}{r^4}(3\cos^2\theta - 1) + \frac{Q^*}{r^2}\right] \quad (16)$$

which has exactly same form with equation (11), obviously one can easily see that the energy contained by the exterior spacetime of the rotating brany black hole in the following form

$$E_{\rm m} = -\frac{|Q^*|}{2r_+}\left(1 + \frac{2a^2}{3r_+^2}\right) . \quad (17)$$

Comparing the expressions (12) and (17) one easily see that the energy associated by the rotating black hole have exactly same form in both Tolman's and Landau-Lifshitz's approach.

5 DISCUSSION

By using a definition of the full field energy of the black hole in the equation (3) one can easily obtain following expression for the gravitational energy of the the rotating brany black hole in the following form Lynden-Bell and Katz (1985); Dadhich and Chellathurai (1986)

$$E_f = M + \frac{|Q^*|}{2r_+}\left(1 + \frac{2a^2}{3r_+^2}\right) , \tag{18}$$

from the equation (18) one can easily see that gravitational energy is positive (i.e., $E_f \geq 0$) for the brany black hole. Introducing the new dimensionless definitions for the brane charge parameter $\beta = |Q^*|/M^2$ and the spin parameter $\alpha = a/M$ of the black hole one can have the expression for the dimensionless gravitational energy in the following form

$$\mathcal{E}_f = \frac{E_f}{M} = 1 + \frac{\beta}{2\left(1 + \sqrt{1 - \alpha^2 + \beta}\right)}\left[1 + \frac{2\alpha^2}{3\left(1 + \sqrt{1 - \alpha^2 + \beta}\right)^2}\right] , \tag{19}$$

In the case of non-rotating brany black hole (i.e., $\alpha = 0$) the gravitational energy will be

$$\begin{aligned} \mathcal{E}_f =& 1 + \frac{\beta}{2\left(1 + \sqrt{1 + \beta}\right)} \\ =& \frac{1}{2}(1 + \sqrt{1 + \beta}) , \end{aligned} \tag{20}$$

For the Kerr black hole (i.e., $\beta = 0$) the dimensionless gravitational energy will be $\mathcal{E}_f = 1$. Figure 3 draws dependence of the gravitational field energy on the brane charge parameter for different values of the spin parameter. One can easily see that the gravitational field energy increase when the brane charge parameter increase.

6 SUMMARY

In the present research we meanly target on the calculation of the energy associated with exterior the rotating brany black hole and its the gravitational energy. We find that the expressions for the energy associated and gravitational energy are exactly the same in Tolman's as well as Landau-Lifshitz's approach. Switching off the brane charge parameter (i.e., $Q^* = 0$) one finds that no energy is contained by the exterior Kerr black hole (i.e., $T^{00} = 0$) and hence the entire energy is confined to its interior only. Although we have evaluated the energy neglecting the terms containing the spin parameter a beyond its third power. We construct the following heuristic arguments in favor of our conclusion. Firstly, once one starts with a definition of energy which demands the entire energy of a Schwarzschild black hole confined to its interior only, we argue that merely the introduction of an intrinsic rotation parameter cannot cause its exterior to share its energy as one must remember that the black hole under investigation is isolated, having no medium surrounding it to propagate its rotational energy to its exterior. Secondly, the rotational energy of the system is proportional to a and in this calculation we have retained terms up to a^3 . So if at all the energy would have been shared by the exterior of the Kerr black hole, it should have certainly appeared in our result. The energy associated with the spacetime of

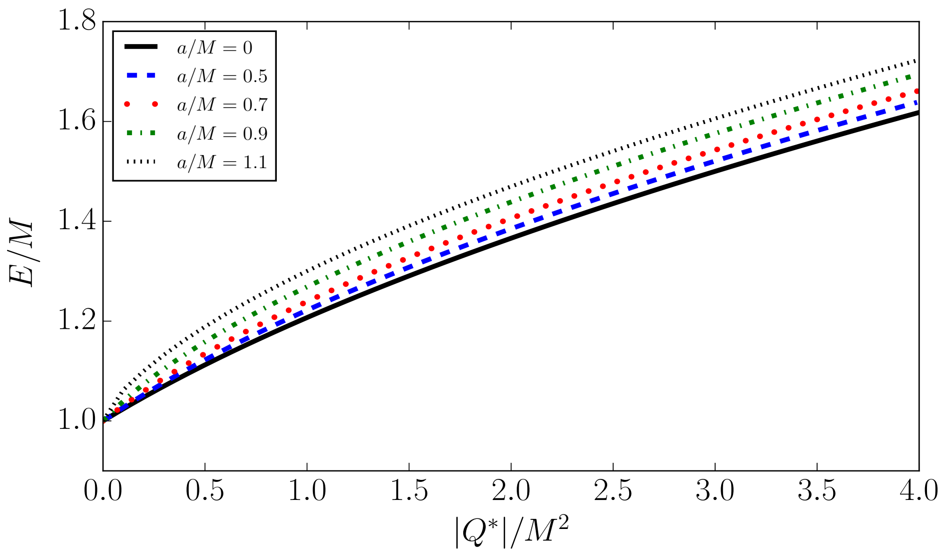

Figure 3. The associated energy dependence of the brane parameter.

the brany black hole of the ($a = 0$) is shared by its interior as well as exterior. Thus one finds a remarkable distinction that Tolman's as well as Landau-Lifshitz's definitions of energy provide between a gravitational field and an electromagnetic field. The entire energy of a gravitational field is confined to its interior only while for the latter case the energy is shared by its interior as well as exterior. In the case of nonrotating black hole (i. e., $a = 0$) we get the energy associated with the spacetime of the brany black hole like metric which is similar the same as found in Virbhadra (1990).

ACKNOWLEDGEMENTS

B.T. thanks B. Ahmedov for very useful comments and discussions. This research is supported by Grant No. VA-FA-F-2- 008, of the Uzbekistan Agency for Science and Technology. This research is partially supported by an Erasmus+exchange grant between SU and NUUz.

REFERENCES

Dadhich, N. and Chellathurai, V. (1986), On gravitational field energy of a charged black hole, *Mon. Not. R. Astron. Soc.*, **220**, pp. 555–558.

Dadhich, N., Maartens, R., Papadopoulos, P. and Rezania, V. (2000), Black holes on the brane, *Physics Letters B*, **487**, pp. 1–6, arXiv: hep-th/0003061.

Landau, L. D. and Lifshitz, E. M. (2004), *The Classical Theory of Fields, Course of Theoretical Physics, Volume 2*, Elsevier Butterworth-Heinemann, Oxford.

Lynden-Bell, D. and Katz, J. (1985), Gravitational field energy density for spheres and black holes, *Mon. Not. R. Astron. Soc.*, **213**, pp. 21P–25P.

Randall, L. and Sundrum, R. (1999), Large Mass Hierarchy from a Small Extra Dimension, *Physical Review Letters*, **83**, pp. 3370–3373, arXiv: hep-ph/9905221.

Tolman, R. C. (1930), On the Use of the Energy-Momentum Principle in General Relativity, *Physical Review*, **35**, pp. 875–895.

Turimov, B. V., Ahmedov, B. J. and Hakimov, A. A. (2017), Stationary electromagnetic fields of slowly rotating relativistic magnetized star in the braneworld, *Physical Review D*, **96**(10), 104001.

Virbhadra, K. S. (1990), Energy associated with a Kerr-Newman black hole, *Physical Review D*, **41**, pp. 1086–1090.