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# Charged particle dynamics in the vicinity of Reissner-Nordström black hole

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**ABSTRACT**
The charged particle motion in Reissner-Nordström spacetime has been discussed. Equation of motion for charged particle around electrically (ECBH) and magnetically (MCBH) charged black holes has been considered by using the Hamilton-Jacobi formalism. The parameters of the innermost stable circular orbit (ISCO) such as energy, angular momentum, and position of particle have been explicitly investigated. The dependences of the energy efficiency, the velocity of charged particle, and the capture cross-section of charged particle by RN black hole on the charge coupling parameter are shown.

**Keywords:** Reissner-Nordström spacetime – black hole – charged particle

## 1 INTRODUCTION

New data from observations of black holes provide new motivations for studying the dynamics of particles in the framework of the General relativity. Because the motion of the test particles clearly describes the properties of the spacetime of the black hole and the hidden singularity. The motion of uncharged and spinless sample particles is controlled only by geodesic equations and directly determines the geodesic structure of spacetime. However, charged test particles can experience not only gravitational, but also an electromagnetic field, and accordingly can provide information about the electromagnetic properties of a black hole. In this work, we studied the circular motion of charged test particles around spherically symmetric, electrically and magnetically charged non-rotating black holes. By using the Hamilton-Jacobi formalism, we obtain the basic equations governing the innermost stability of circular orbits and the associated energies, angular moments, and also the particle velocities in these orbits.

The detailed analyses of neutral particle motion Pugliese et al. (2011a), dynamics charged of particle Bini et al. (2007); Pugliese et al. (2011b, 2017); Das et al. (2017) in Reissner-Nordström spacetime has been studied. In the Ref. Grunau and Kagramanova

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(2011) geodesics of electrically and magnetically charged test particles in the Reissner-Nordström spacetime has been investigated. The capture cross-section of massless and massive particles by the charged black hole has been investigated in Zakharov (1994). The innermost stable circular orbits of charged spinning test particles have been analyzed in Zhang and Liu (2019). In Ref. Zaslavskii (2010) the effect of charged particles acceleration by the black holes in Reissner-Nordström spacetime has been studied. In Refs. Zajaček et al. (2018); Ghosh et al. (2020) the significance of the electric and magnetic charge of the astrophysical black hole has been discussed. In Ref. Stuchlík et al. (2020) dynamical motion of charged particle in the vicinity of rotating black hole in the presence of external magnetic field has been investigated.

In the present research, we are interested in investigating of charged particle motion around electrically (ECBH) and magnetically (MCBH) charged black holes. One of the simple candidates for such black holes is described by the Reissner-Nordström metric as

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{1}$$

where $M$ is the gravitational mass of a black hole, $Q$ is its total charge which can be either the *electric charge* ($Q = Q_e$) or *magnetic charge* ($Q = Q_m$). In these cases, the components of the associated vector potential of the electromagnetic fields are given as

$$A_t = -\frac{Q_e}{r}, \qquad A_\phi = Q_m \cos\theta, \tag{2}$$

Note that the metric (1) together with (2) are, fully, satisfied the Einstein-Maxwell field equations. The radius of outer spacelike horizon can be calculated, from equation $f = 0$, as $r_+ = M^2 + \sqrt{M^2 - Q^2}$. Notice that throughout the paper, we use system of a geometrized units, $G = c = 1$.

## 2 HAMILTON-JACOBI EQUATION

The Hamilton-Jacobi equation for charged particle of mass $m$ and charge $q$ is given by

$$g^{\alpha\beta}\left(\frac{\partial S}{\partial x^\alpha} - qA_\alpha\right)\left(\frac{\partial S}{\partial x^\beta} - qA_\beta\right) = -m^2, \tag{3}$$

with solution, $S = -Et + L\phi + S_r + S_\theta$, then

$$-\frac{1}{f}\left(E + \frac{qQ_e}{r}\right)^2 + f\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\theta}{\partial\theta}\right)^2 + \frac{(L - qQ_m\cos\theta)^2}{r^2\sin^2\theta} = -m^2\,. \tag{4}$$

where $E$, $L$ are the energy and angular momentum of test particle at the infinity, respectively. $S_r$ and $S_\theta$ are the radial and angular functions. Here one can see that equation (4) is fully separable into radial and angular parts. Hereafter performing simple algebraic manipulations, one can show that

$$S_r = \int \frac{\mathrm{d}r}{f}\sqrt{\left(E + \frac{qQ_e}{r}\right)^2 - f\left(m^2 + \frac{K}{r^2}\right)}, \tag{5}$$

$$S_\theta = \int \mathrm{d}\theta \sqrt{K - \frac{(L - qQ_m\cos\theta)^2}{\sin^2\theta}}, \tag{6}$$

where $K$ is the Carter constant of motion.

Before go further, we introduce the following useful notations:

$$\mathcal{E} = \frac{E}{m}, \quad \mathcal{L} = \frac{L}{mM}, \quad \mathcal{K} = \frac{K}{(mM)^2}, \quad \mathcal{Q} = \frac{Q}{M}, \tag{7}$$

and the radial coordinate is normalized as $r \to r/M$. Now we write components of momentum as $p^\alpha = g^{\alpha\beta}(\partial S/\partial x^\beta)$, on the other hand $p^\alpha = m\dot{x}^\alpha = m(dx^\alpha/d\lambda)$, where $\lambda$ is an affine parameter. Finally, taking into account all facts above, equations of motion can be written as

$$\dot{t} = \frac{1}{f}\left(\mathcal{E} + \frac{\sigma_e}{r}\right), \qquad \dot{\phi} = \frac{\mathcal{L} - \sigma_m\cos\theta}{r^2\sin^2\theta}, \tag{8}$$

$$\dot{r}^2 = \left(\mathcal{E} + \frac{\sigma_e}{r}\right)^2 - f\left(1 + \frac{\mathcal{K}}{r^2}\right) \equiv \frac{R(r)}{r^4}, \qquad R(r) \geq 0, \tag{9}$$

$$\dot{\theta}^2 = \frac{1}{r^4}\left[\mathcal{K} - \frac{(\mathcal{L} - \sigma_m\cos\theta)^2}{\sin^2\theta}\right] \equiv \frac{T(\theta)}{r^4}, \qquad T(\theta) \geq 0, \tag{10}$$

where the charge coupling parameters are defined as

$$\sigma_e = \frac{qQ_e}{mM}, \qquad \sigma_m = \frac{qQ_m}{mM}\ . \tag{11}$$

According to Refs. Shapiro and Teukolsky (1983); Misner et al. (1973), the spatial components of velocity of particle measured by a local observer can be determined as

$$v_{\hat{r}} = \sqrt{-\frac{g_{rr}}{g_{tt}}\frac{\mathrm{d}r}{\mathrm{d}t}} = \sqrt{1 - f\frac{r^2 + \mathcal{K}}{(r\mathcal{E} + \sigma_e)^2}}, \tag{12}$$

$$v_{\hat{\theta}} = \sqrt{-\frac{g_{\theta\theta}}{g_{tt}}\frac{\mathrm{d}\theta}{\mathrm{d}t}} = \frac{\sqrt{f}}{r\mathcal{E} + \sigma_e}\sqrt{\mathcal{K} - \frac{(\mathcal{L} - \sigma_m\cos\theta)^2}{\sin^2\theta}}, \tag{13}$$

$$v_{\hat{\phi}} = \sqrt{-\frac{g_{\phi\phi}}{g_{tt}}\frac{\mathrm{d}\phi}{\mathrm{d}t}} = \frac{\sqrt{f}}{r\mathcal{E} + \sigma_e}\frac{\mathcal{L} - \sigma_m\cos\theta}{\sin\theta}, \tag{14}$$

which allows to write

$$\mathcal{E} = \frac{\sqrt{f}}{\sqrt{1 - v^2}} - \frac{\sigma_e}{r}, \qquad v^2 = v_{\hat{r}}^2 + v_{\hat{\theta}}^2 + v_{\hat{\phi}}^2\ . \tag{15}$$

Notice that near the horizon i.e. $f = 0$, the radial velocity will be $v_{\hat{r}} = 1$, while angular componets vanish $v_{\hat{\theta}} = v_{\hat{\phi}} = 0$.

## 3 INNERMOST STABLE CIRCULAR ORBIT (ISCO)

It is important to compute the radius of the stable circular orbit of test particle so-called innermost stable circular orbit (ISCO) radius. In order to find the ISCO radius for test particle one can use the following conditions:

$$R(r) = 0, \qquad \frac{\mathrm{d}R(r)}{\mathrm{d}r} = 0, \qquad \frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} = 0, \tag{16}$$

$$T(\theta) = 0, \qquad \frac{\mathrm{d}T(\theta)}{\mathrm{d}\theta} = 0, \qquad \frac{\mathrm{d}^2T(\theta)}{\mathrm{d}\theta^2} = 0, \tag{17}$$

where the first equations in (16) and (17) provide the particle motion to be in the circular orbit (i.e. $\dot{r} = \dot{\theta} = 0$), while the first order derivatives with respect to coordinates $(r, \theta)$ represent the stationary points of the functions $R(r), T(\theta)$. Finally, the last conditions in (16) and (17) correspond the minimum of the radial and angular functions. For simplicity, assume that black hole charge is negligibly small to change background spacetime, ($Q^2 \to 0$, so that the lapse function will be $f(r) = 1-2M/r$), however, contribution of the interaction terms are large enough in dynamics particle orbiting around black hole.

### 3.1 Charged particle in the vicinity MCBH

We first discuss charged particle motion around MCBH with $\sigma_m \neq 0$ and $\sigma_e = 0$. Hereafter using the conditions (17), we obtain

$$\theta_0 = \tan^{-1}\left(\frac{\sigma_m}{\mathcal{L}}, \pm\frac{\sqrt{\mathcal{L}^2 - \sigma_m^2}}{\mathcal{L}}\right), \qquad \mathcal{K} = \mathcal{L}^2 - \sigma_m^2 \,. \tag{18}$$

Then after eliminating the Carter constant, the radial function takes a form:

$$R(r) = \left(\mathcal{E}^2 - 1\right) r^4 + 2r^3 - \left(\mathcal{L}^2 - \sigma_m^2\right) r^2 + 2\left(\mathcal{L}^2 - \sigma_m^2\right) r, \tag{19}$$

Recalling the conditions (16), and performing simple algebraic manipulations, one can obtain

$$\mathcal{E} = f\sqrt{\frac{r}{r-3}}, \qquad \mathcal{L} = \sqrt{\frac{r^2}{r-3} + \sigma_m^2}, \qquad r = 6, \tag{20}$$

Finally, parameters of the ISCO such as the energy $\mathcal{E}_0$, angular momentum $\mathcal{L}_0$, angle $\theta_0$ and radius $r_0$ for charged particle orbiting around MCBH take the form:

$$\mathcal{E}_0 = \frac{2\sqrt{2}}{3}, \quad \mathcal{L}_0 = \sqrt{12 + \sigma_m^2}, \quad \theta_0 = \tan^{-1}\left(\pm\frac{2\sqrt{3}}{\sigma_m}\right), \qquad r_0 = 6 \,. \tag{21}$$

As one can see from equation (21) a position of the ISCO for charged particle orbiting around MCBH is located at $r_{\rm ISCO} = 6$ and $\theta_{\rm ISCO} = \tan^{-1}\left(2\sqrt{3}/\sigma_m\right)$. Figure 1 draws dependence of ISCO positions from the charge coupling parameter.

On the other hand, it is easy to show that the orbital velocity measured by the local observer at the ISCO will be independent of coupling parameter $\sigma_m$, and it equals half of the speed of light, i.e. $v = 1/2$ as for neutral particle.

It is also an interesting task to demonstrate capture cross-section of charged particle by the black hole. According to Ref. Zakharov (1994), the impact parameter of a massive particle, $b$, can be found as,

$$b^2 = \alpha \mathcal{L}^2, \qquad \alpha = \frac{1}{\mathcal{E}^2 - 1}, \tag{22}$$

which allows to determine capture cross section $\sigma = \pi b^2$. Taking into account definition (22), the radial functions can be rewritten as, $\mathcal{R} = \alpha R/r$, or

$$\mathcal{R}(r) = r^3 + 2\alpha r^2 - \left(b^2 - \alpha\sigma_m^2\right) r + 2\left(b^2 - \alpha\sigma_m^2\right), \tag{23}$$

The existence condition for multiple roots is equivalent to vanishing of the discriminant of cubic equation (23), after simple calculations, one can obtain the explicit expressions for the impact parameter of charged particle orbiting around MCBH .... in the form:

$$b^2 = \frac{1}{2}\left[27 + 18\alpha - \alpha^2 + (\alpha + 9)\sqrt{(\alpha + 9)(\alpha + 1)}\right] + \alpha\sigma_m^2, \tag{24}$$

In comparison with the expression for impact parameter in Ref. Zakharov (1994), there is additional term in equation (24) given as $\alpha\sigma_m^2$ which arises due to the electromagnetic interaction between the black hole and particle. From here one can conclude that capture cross-section of charged particle by MCBH increases for $\alpha > 0$, while it decreases for $\alpha < 0$ in comparison with that for a neutral particle.

### 3.2 Charged particle in the vicinity ECBH

Now we focus on charged particle motion around ECBH with $\sigma_e \neq 0$, $\sigma_m = 0$. Again after using the conditions in (17), one can find that particle is located in an equatorial plane with $\theta_0 = \pi/2$ and $\mathcal{K} = \mathcal{L}^2$. Then the radial function takes a form:

$$R(r) = \left(\mathcal{E}^2 - 1\right) r^4 + 2\left(1 - \mathcal{E}\sigma_e\right) r^3 - \left(\mathcal{L}^2 - \sigma_e^2\right) r^2 + 2\mathcal{L}^2 r, \tag{25}$$

Hereafter using conditions (16), one can obtain

$$\mathcal{E} = f\sqrt{\frac{r}{r-3} + \frac{\sigma_e^2}{4(r-3)^2}} - \frac{1}{2}\sigma_e \frac{r-4}{r(r-3)}, \tag{26}$$

$$\mathcal{L}_\pm^2 = \frac{r^2}{r-3}\left[1 \pm \sigma_e f\sqrt{\frac{r}{r-3} + \frac{\sigma_e^2}{4(r-3)^2}} + \frac{1}{2}\sigma_e^2 f \frac{1}{r-3}\right], \tag{27}$$

and

$$\left(r^2 - 4r + 6\right)\sigma_e^2 \pm (r-6)(r-2)\sigma_e\sqrt{4r(r-3) + \sigma_e^2} - 2r(r-3)(r-6) = 0, \tag{28}$$

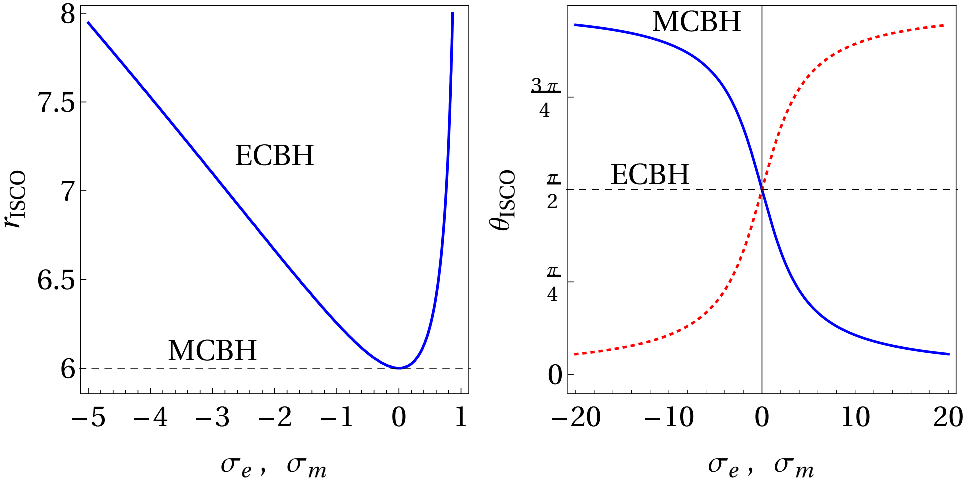


**Figure 1.** Dependence of the ISCO positions for charged particle orbiting around ECBH and MCBH on the charge coupling parameters $(\sigma_e, \sigma_m) \equiv (Q_e, Q_m)(q/mM)$.

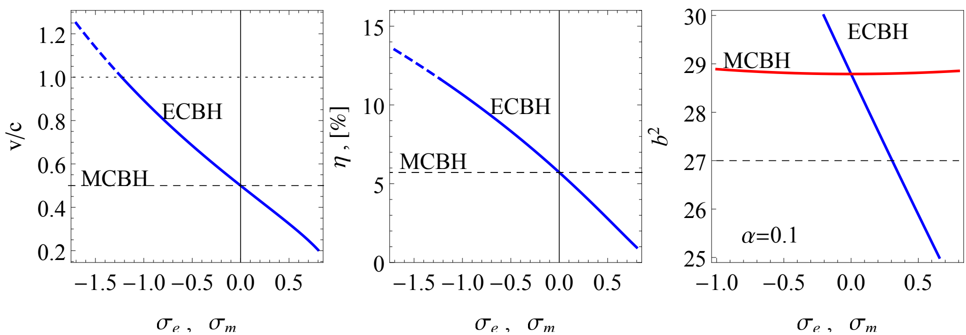


**Figure 2.** Dependence of the orbital velocity (left panel), the energy efficiency (centeral panel) and the impact parameter (right panel) of charged particle orbiting around ECBH and MCBH on the charge coupling parameters $(\sigma_e, \sigma_m) \equiv (Q_e, Q_m)(q/mM)$.

which allows finding the ISCO radius for charged particle. Unfortunatly, it is difficult to find analytical solution of the equation (28). However, careful numerical analyzes show that the ISCO for charged particle orbiting around will be always greater than that for neutral particle. Figure 1 draws dependence of the ISCO radius on the charge coupling parameters $(\sigma_e, \sigma_m)$.

The orbital velocity measured by the local observer at the ISCO will be dependent on coupling parameter $\sigma_e$, and this dependence is expressed as follows:

$$v = \frac{\sqrt{f}\mathcal{L}}{r\mathcal{E} + \sigma_e} \,. \tag{29}$$

Figure 2 illustrates the dependence of the orbital velocity of charged particle measured by a local observer at the ISCO on the coupling parameter $\sigma_e$. It shows that for the positive value of the coupling parameter orbital velocity at the ISCO will be smaller than a half of the speed of light, $v = 1/2$, (which is responsible for neutral particle), and it decreases up to value $v \sim 0.16$, while for a negative value of coupling parameter it increases almost linearly and becomes larger than the speed of light for $\sigma_e < -1.23$, which corresponds to so-called superluminal motion. However, from the physical point of view, a massive particle can not move faster than the speed of light that is why at a sudden value of velocity charged particle starts to radiate and loses its energy.

In order to determine capture cross-section of particle by ECBH, we again write the radial function in the form:

$$\mathcal{R}(r) = r^3 + 2\left(\alpha - \sigma_e\sqrt{\alpha(1+\alpha)}\right)r^2 - \left(b^2 - \alpha\sigma_e^2\right)r + 2b^2 \ . \tag{30}$$

As we mentioned before that the discriminant of the equation above should vanishes. Since the expression for the impact parameter is not simple, we decided to solve numerically. Figure 2 shows the dependence of the impact parameter on the charge coupling parameter for the case when $\alpha$ is equal to 0.1. The graph shows that the capture cross-section of charged particle by ECBH strongly depends on the charge coupling parameter, and increases with decreasing it.

### 3.3 Energy efficiency

It is also interesting to analyze the energy efficiency of the test particle, the ratio of the binding energy $mc^2 - E_{\rm ISCO}$, and the rest energy $mc^2$, can be found as, $\eta = 1 - \mathcal{E}_0$. As we show before that the ISCO energy for charged particle orbiting around MCBH is the same as for neutral particle, which means the energy efficiency should be the same. So that one can obtain $\eta_{\rm MCBH} \sim 6\%$. On other hand, the detailed analyses show that the energy efficiency for charged particle orbiting around ECBH strongly depends on the charged coupling parameter $\sigma_e$, that can reach a maximal value of $\sim 12\%$ as shown in Fig.2.

## 4 CONCLUSIONS AND FUTURE OUTLOOK

We investigated charged test particle motion in Reissner-Nordstrom spacetime. Using the Hamiltonian formalism, equation of motion for charged particle orbiting around both ECBH and MCBH has been explicitly derived. It is shown that the charge coupling parameter dramatically changes the behavior of particle in the vicinity of the black hole.

The parameters of the innermost stable circular orbit (ISCO) such as specific energy, specific angular momentum, and position of particle have been explicitly discussed. It is shown that the ISCO position for charged particle is located in an equatorial plane and will be greater than that for a neutral particle in the vicinity of ECBH, while in the vicinity of MCBH, it is the same as that for a neutral particle, but displaced from the equatorial plane. It is also shown that charged particle at the ISCO moves around MCBH with half of the speed of light, independently from the coupling parameter $\sigma_m$, like a neutral particle. However, the orbital velocity of charged particle moving at the ISCO around ECBH strongly depends on the charge coupling constant. It increases with decreasing couple parameter, even becomes than the speed of light for the value $\sigma_e < -1.23$, which corresponds to so-called superluminal motion. However, from the physical point of view, a massive particle can not move faster than the speed of light that is why at a sudden value of velocity charged particle stars to radiate and loses its energy.

It has been shown that the dependence of capture cross-section on the charge coupling parameter, it can be seen that capture cross-section of charged particle by MCBH depends on square of the charge coupling parameter, and it decreases by decreasing $\alpha$ parameter. But, unlike the case for MCBH, the capture cross-section of charged particle by ECBH depends from the charge coupling parameter, and detailed analyses showed that this dependence will be stronger by increasing of the energy parameter $\alpha$.

It is also shown that the energy efficiency of charged particle orbiting around MCBH will be independent of charge parameter, i.e $\eta_{\rm MCBH} \simeq 6\%$, while in the vicinity of ECBH, it strongly depends on charged parameter, $\sigma_e$, that can reach a maximal value of $\eta_{\rm ECBH} \sim 12\%$.

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