Analytical solution for charged fluid pressure profiles - circulation in combined electromagnetic field

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ABSTRACT
We introduce a general transformation leading to an integral form of pressure equations characterizing equilibrium configurations of charged perfect fluid circling in strong gravitational and combined electromagnetic fields. The transformation generalizes our recent analytical treatment applicable to electric or magnetic fields treated separately along with the gravitational one. As an example, we present a particular solution for a fluid circling close to a charged rotating black hole immersed in an asymptotically uniform magnetic field.

Keywords: Charged perfect fluid – toroidal structures – compact objects – spacetime – electromagnetic field – pressure equations – unified integral form – Wald configuration – polar clouds – covering shells

1 INTRODUCTION

Investigation of fluids under astrophysical conditions represents one of the most challenging tasks in physics. Encircling sources of strong gravity, such as black holes and neutron stars (compact objects), astrophysical fluids manifest as compressible (gaseous) fluids, typically. We can find them in a form of pure neutral or quasi–neutral ionized gas (plasma), as a neutral or charged microscopic dust, or dust grains (pressure–less fluid), and very often as a dispersed medium, such as dusty gas, dusty plasma, etc. The gaseous fluids can range from extremely diluted ones (represented by separated particles, and described within the test–particles approach), through diluted ones (described within the kinetic approach), up to dense fluids (conveniently studied within the magneto–hydrodynamic approach).

From another point of view, astrophysical fluids can whirl in extremely complex dynamical situations, for instance, falling down and accreting onto compact objects, or being

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launched in the form of winds or jets; on the other hand, fluids can settle down in equilibrium configurations and circle around compact objects.

In recent years, we have been focusing on the latter scenario, i.e on the magneto–hydrodynamic study of electrically charged fluids forming equilibrium toroidal–like structures around compact objects. For this purpose, in a serie of papers, we developed a relevant basic general relativistic (Kovář et al., 2011, 2014, 2016) and Newtonian (Slaný et al., 2013; Trova et al., 2016) models, and introduced them in several gravito–electromagnetic backgrounds, revealing interesting configurations of the charged fluid. For instance, in contrast to the neutral perfect fluid being capable to form toroidal structures only in the equatorial plane (Kozłowski et al., 1978; Abramowicz et al., 1978), the charged fluid circling in a proper electromagnetic background can soar up the equatorial plane and form a 'levitating' torus, or it can 'hover' above the compact object as a 'polar cloud'.

The toroidal structures are well determined by their pressure profiles as solutions of the coupled pressure differential equations – the fundamental equations characterizing the considered model. These pressure equations, being accompanied by an integrability condition and equations of state, can be satisfactorily solved in a numerical way. The analytical treatment, however, is more traditional, enabling easier subsequent processing and bringing a deeper insight into the studied problem.

Here, we focus on the general relativistic approach, and introduce a transformation leading to an unified integral form of the pressure equations characterizing the charged fluid motion in more general background than considered up to now. The transformation enables us to analytically treat the fluid circulation in gravitational field being accompanied by a combined electromagnetic field, i.e. by the field characterized by two non–zero components of its vector potential.

2 MODEL IN DIFFERENTIAL DESCRIPTION

The considered model of charged fluid circling in strong gravitational and electromagnetic fields can be characterized by the following assumptions: 1) pure azimuthal circulation of the fluid with elementary charges adherent to the moving fluid components, 2) gravito–electromagnetic test fluid, and axial symmetry and stationarity of the gravito–electromagnetic background, 3) fluid properties to be of the perfect fluid one and satisfying the polytropic pressure–density relation; utilization of these assumptions within the general relativistic magneto–hydrodynamic equations (conservation laws, Maxwell equations and Ohm law) provides us with the basic coupled pressure equations.

2.1 Pressure equations

The rotating fluid with profiles of charge density q_ρ and total energy density ϵ forms a structure being determined by the iso–surfaces of the pressure p (equi–pressure surfaces), which can be determined from the coupled pressure equations

$$\begin{aligned} \partial_r p &= -(p+\epsilon)\mathbb{R}_1 + q_\rho \mathbb{R}_2 \equiv \mathbb{R}, \qquad (1)\\ \partial_\theta p &= -(p+\epsilon)\mathbb{T}_1 + q_\rho \mathbb{T}_2 \equiv \mathbb{T}, \end{aligned}$$

where $\mathbb{R} = \mathbb{R}(r, \theta)$ and $\mathbb{T} = \mathbb{T}(r, \theta)$ denote the right hand sides of these equations, and

$$\mathbb{R}_1 = \partial_r \ln|U_t| - \frac{\omega \partial_r \ell}{1 - \omega \ell}, \quad \mathbb{R}_2 = U^t \partial_r A_t + U^\phi \partial_r A_\phi, \tag{2}$$
$$\mathbb{T}_1 = \partial_\theta \ln|U_t| - \frac{\omega \partial_\theta \ell}{1 - \omega \ell}, \quad \mathbb{T}_2 = U^t \partial_\theta A_t + U^\phi \partial_\theta A_\phi.$$

Here, the electromagnetic vector potential has the t and ϕ independent form $A_\alpha = (A_t, 0, A_\phi, 0)$ in the coordinate system (t, r, ϕ, θ), $U^\alpha = (U^t, 0, U^\phi, 0)$ is the fluid 4-velocity, $\ell = -U_\phi/U_t$ the specific angular momentum and $\omega = U^\phi/U^t$ the the angular velocity, all related by the formulae

$$\omega = -\frac{\ell g_{tt} + g_{t\phi}}{\ell g_{t\phi} + g_{\phi\phi}}, \quad (U_t)^2 = \frac{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}{\ell^2 g_{tt} + 2\ell g_{t\phi} + g_{\phi\phi}}, \tag{3}$$

in a prescribed gravitational field $g_{\alpha\beta}$. Note that a derivation of the pressure equations (1) can be found in (Kovář et al., 2011, 2014, 2016); their uncharged limit $q_\rho = 0$ corresponds to the Euler equations describing a rotating electrically neutral perfect fluid (Kozłowski et al., 1978; Abramowicz et al., 1978).

2.2 Rotation regime, charge distribution and thermodynamic setup

The pressure equations (1) are not generally integrable and must be accompanied by the integrability condition

$$\partial_\theta \mathbb{R} + \mathbb{T} \partial_p \mathbb{R} = \partial_r \mathbb{T} + \mathbb{R} \partial_p \mathbb{T}. \tag{4}$$

From the physical point of view, this condition relates the charge density distribution $q_\rho = q_\rho(r, \theta)$ throughout the torus and its rotation regime $\ell = \ell(r, \theta)$ (or $\omega(r, \theta)$), which must be properly adjusted to each other according to this condition.

Formally, to close the system of equations, it is necessary to specify relations for the pressure and total energy density. For the purpose of basic theoretical investigation, we can consider a compressible perfect fluid satisfying the general polytropic equation of state for the pressure

$$p = \kappa \rho^\Gamma, \tag{5}$$

together with the total energy density relation

$$\epsilon = \rho + \frac{1}{\Gamma - 1} p, \tag{6}$$

with κ and Γ being the polytropic coefficient and exponent, and ρ the rest–mass density.

3 UNIQUE SOLUTION

In order to avoid numerical integrations of the pressure equations (1) and the related integrability condition (4), we can introduce a transformation of the charge density together with a transformation of the pressure. Then under certain conditions specified below, the system of differential pressure equations can be uniquely rewritten, unified and integrated.

3.1 Transformation of pressure equations and correction function

Defining the charge density transformation by the relation

$$K = \frac{q_\rho}{\epsilon + p}, \tag{7}$$

where we address the function K as the 'correction function' (on the basis of integrability condition (4) mathematically ensuring the integrability of the pressure equations (1) after the profiles $\ell(r,\theta)$ or $\omega(r,\theta)$ are set), and the pressure transformation by the coupled equations

$$\partial_r h = \frac{\partial_r p}{(p+\epsilon)}, \quad \partial_\theta h = \frac{\partial_\theta p}{(p+\epsilon)}, \tag{8}$$

where we address the function h as the 'auxiliary function', we get the system of transformed pressure equations in the form

$$\begin{aligned} \partial_r h &= -(\mathbb{R}_1 - K\mathbb{R}_2), \\ \partial_\theta h &= -(\mathbb{T}_1 - K\mathbb{T}_2), \end{aligned} \tag{9}$$

accompanied by the integrability condition

$$\partial_\theta(\mathbb{R}_1 - K\mathbb{R}_2) = \partial_r(\mathbb{T}_1 - K\mathbb{T}_2). \tag{10}$$

Providing that $\epsilon = \epsilon(p)$, which is guaranteed by the chosen thermodynamic alignment (5)-(6), the auxiliary function can be explicitly expressed as

$$h = \int_0^h \mathrm{d}h = \int_0^p \frac{\mathrm{d}p}{p+\epsilon} = \ln\left(1 + \frac{\Gamma\kappa^{\frac{1}{\Gamma}} p^{\frac{\Gamma-1}{\Gamma}}}{\Gamma - 1}\right). \tag{11}$$

In the considered combined electromagnetic field, it is, moreover, very convenient to rescale the correction functions as

$$\mathcal{K} = KU^\phi. \tag{12}$$

Then, our system of the transformed pressure equations (9) can be written in the form

$$\begin{aligned} \partial_r h &= -\partial_r \ln|U_t| + \frac{\omega\partial_r\ell}{1-\omega\ell} + \mathcal{K}(\omega^{-1}\partial_r A_t + \partial_r A_\phi), \\ \partial_\theta h &= -\partial_\theta \ln|U_t| + \frac{\omega\partial_\theta\ell}{1-\omega\ell} + \mathcal{K}(\omega^{-1}\partial_\theta A_t + \partial_\theta A_\phi), \end{aligned} \tag{13}$$

indicating a possible unification. After an introduction of the electromagnetic vector time–component transformation according to the relations

$$\partial_r a_t = \omega^{-1}\partial_r A_t, \quad \partial_\theta a_t = \omega^{-1}\partial_\theta A_t, \tag{14}$$

and the 'magnetic function' $A = a_t + A_\phi$, we get the final system of the transformed pressure equations

$$\partial_r h = -\partial_r \ln|U_t| + \frac{\omega \partial_r \ell}{1-\omega\ell} + \mathcal{K}\partial_r A, \tag{15}$$
$$\partial_\theta h = -\partial_\theta \ln|U_t| + \frac{\omega \partial_\theta \ell}{1-\omega\ell} + \mathcal{K}\partial_\theta A.$$

3.2 Solution for the h-function

Providing that $\omega = \omega(\ell)$ and $\mathcal{K} = \mathcal{K}(A)$, on one hand restricting degrees of freedom in the model, but still providing realistic physical scenarios, we can join equations of the system (15) into the unified integral form

$$\int_0^h \mathrm{d}h = -\ln\left|\frac{U_t}{U_{t_\mathrm{in}}}\right| + \int_{\ell_\mathrm{in}}^{\ell} \frac{\omega \mathrm{d}\ell}{1-\omega\ell} + \int_{A_\mathrm{in}}^{A} \mathcal{K}\mathrm{d}A, \tag{16}$$

with the solution written in the closed form

$$h = -H + H_\mathrm{in}. \tag{17}$$

Here, the function $H(r,\theta)$ represents the variable part (potential) in the right–hand side of equation (16) after the integration, and the subscript 'in' refers to the inner edge of the structure at $r = r_\mathrm{in}$ and $\theta = \theta_\mathrm{in}$, determining the constants of integration being coupled in H_in. Then, thanks to the transformation (11), equi–pressure surfaces $p = \mathrm{const}$ determining the topology of the fluid structure take the same shapes as equi–potential surfaces $H = \mathrm{const}$.

3.3 Solution in Wald configuration

As an example, we can present our charged fluid structures circling close to a rotating charged black hole immersed in an asymptotically uniform magnetic field. Such a background can be advantageously described by the Wald test–field solution of Maxwell equations (Wald, 1974) in the Kerr spacetime. In the standard general relativistic dimensionless units, it reads

$$A_t = \tfrac{1}{2}\,B\Big(g_{t\phi} + 2a\,g_{tt}\Big) - \tfrac{1}{2}\,Q\,g_{tt} - \tfrac{1}{2}\,Q, \tag{18}$$
$$A_\phi = \tfrac{1}{2}\,B\Big(g_{\phi\phi} + 2a\,g_{t\phi}\Big) - \tfrac{1}{2}\,Q\,g_{t\phi}, \tag{19}$$

where the charge Q and magnetic field strength B parameters are only test-field parameters, thus not influencing the background spacetime geometry $g_{\alpha\beta}$.

By setting the rigid rotation regime of the fluid, i.e. $\omega = \mathrm{const}$, the electromagnetic vector time–component transformation can be chosen in the form $a_t = \omega^{-1}A_t$, and the corresponding magnetic function will take the form $A = \omega^{-1}A_t + A_\phi$. Next, for the purpose of a basic illustration, even if we choose the rescaled correction function with a very simple profile, such as

$$\mathcal{K} = \mathcal{K}(A) = kA, \tag{20}$$

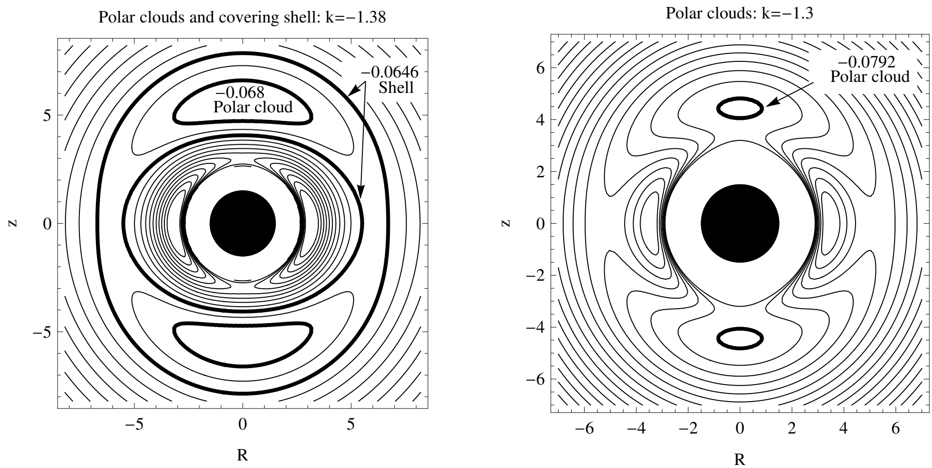

Figure 1. Two examples of typical behavior of the potential $H(r, \theta)$ shown in terms of the poloidal equi–potential contours plotted in the cylindrical coordinates $R = r \sin\theta$, $z = r \cos\theta$. Particularly, we present topologies embodying two polar potential minima surrounded by closed equi–potential surfaces (polar clouds) and two equatorial maxima, corresponding to the background and fluid parameters $a = 0.9$, $Q = 3.99 \times 10^{-3}$, $B = 6.5 \times 10^{-5}$, $\omega = -1.6 \times 10^{-3}$, and $k = -1.38$ and $k = -1.3$. The topology corresponding to the case $k = -1.38$ (left) embodies also coupled closed equi–potential surfaces (covering shell) all around the central black hole.

where the coefficient k scales the charge profile of the fluid, we reveal very interesting behavior of the related H–potential. The behavior of the potential H shows that along with the typical tori centered and circling in the equatorial plane, the topology of the charged fluid structures can also exhibit structures such as 'polar clouds' or 'covering shells' (see Fig. 1).

4 CONCLUSIONS

The introduced integral form of the pressure equations (16) represents an extremely convenient formula, allowing us to avoid a standard general treatment of coupled partial differential equations; its uncharged limit is referred to as Boyer's condition and useful for investigation of the neutral fluid toroidal structures – the so–called 'Polish doughnuts' (Abramowicz et al., 1978). On the other hand, the restricting conditions for the unification $p = p(\epsilon)$, $\omega = \omega(\ell)$ and $\mathcal{K} = \mathcal{K}(A)$ can prevent us from treating some more general interesting regimes in this way; a numerical integration of the pressure equations is then necessary.

The presented example of the pressure equations unification in the case of the Wald configuration could provide an illustration how efficient the procedure can be, despite of the physical complexity of the studied scenario – the interplay of strong gravity and electromagnetism forming a rotating charged fluid. No doubt, the introduced rigidly rotating polar clouds and covering shells represent sufficient reasoning for a detailed survey of different possible classes of the toroidal topology not only in the Wald configuration. Note that the existence of the polar clouds was already mentioned in (Kovář et al., 2014). There, however, the situation was considered simpler (the background spacetime did not rotate, $a = 0$), and the unification of the pressure equations was not introduced; an analytical solution was obtained by a direct solution of the coupled pressure equations.

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