Charged particle motion around Schwarzschild black hole with split monopole magnetosphere

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ABSTRACT
Charged particle dynamics around Schwarzschild black hole with split monopole magnetic field has been examined. Using an effective potential technique the position of stable circular orbits off- and in-equatorial plane has been found. Equations of motion for charged particle trajectories have been solved numerically and some particle trajectories have been given. Also frequencies for perturbed particle circular orbit has been calculated.

Keywords: charged particle – black holes – split magnetic monopole

1 INTRODUCTION

Weak test magnetic fields will have negligible effect on background spacetime or on the motion of neutral particles. However, for the motion of charged test particles, the influence of the magnetic field on particle dynamics can be really large. For charged test particle with charge q and mass m moving in vicinity of a black hole (BH) with mass M surrounded by magnetic field of the strength B, one can introduce a dimensionless quantity $qBGM/mc^4$ that can be identified as relative Lorenz force. This quantity can be quite large even for weak magnetic fields due to the large value of the specific charge q/m. In our approach the ”charged particle” can represent matter ranging from electron to some charged inhomogeneity orbiting in the innermost region of the accretion disk. The charged particle specific charges q/m for any such structure will then range from the electron maximum to zero.

In this paper we will concentrate our attention on BH magnetosphere given by split monopole solution Blandford and Znajek (1977). The radial profile of the split monopole magnetic field configuration could be relevant for magnetic filed generated by thin accretion disk around BH Komissarov (2004) or also for magnetosphere generated by another BH accretion configurations but close to the BH horizon Komissarov (2005).

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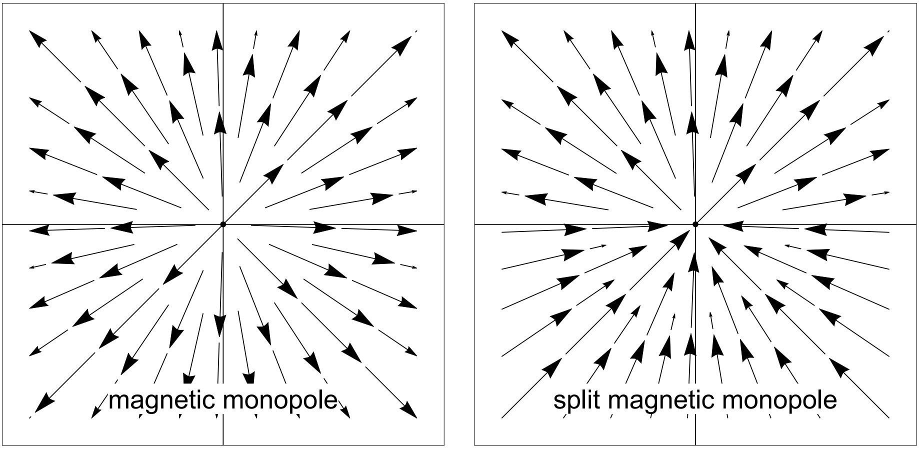

Figure 1. Magnetic field lines for monopole (left) and split monopole (right) solutions.

Throughout the present paper we use the spacelike signature $(-, +, +, +)$, and the system of geometric units in which $G = 1 = c$.

2 CHARGED PARTICLE DYNAMICS

We describe dynamics of charged particle with charge $q \neq 0$ in the vicinity of the Schwarzschild BH embedded in magnetic field, using Hamiltonian formalism Kološ et al. (2015).

The gravity will enter to the equations of motion through Schwarzschild black hole (with mass M) spacetime line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{1}$$

There are no magnetic monopoles in the classical theory of electromagnetism, and for magnetic field we have

$$\mathrm{div}\boldsymbol{B} = 0. \tag{2}$$

The source for magnetic monopole field, artificial magnetic monopole, will be located in coordinate system origin. Split monopole solution is monopole solution, where we change the orientation of magnetic field lines below equatorial plane (III and IV quadrants) and hence the condition 2 will be satisfied, see Fig.1. The source for split monopole magnetic field configuration will be some electric current floating around the coordinate system origin in infinitesimally thin disk located in equatorial plane. In this work, we consider BH magnetosphere in the form of split monopole solution Blandford and Znajek (1977). The

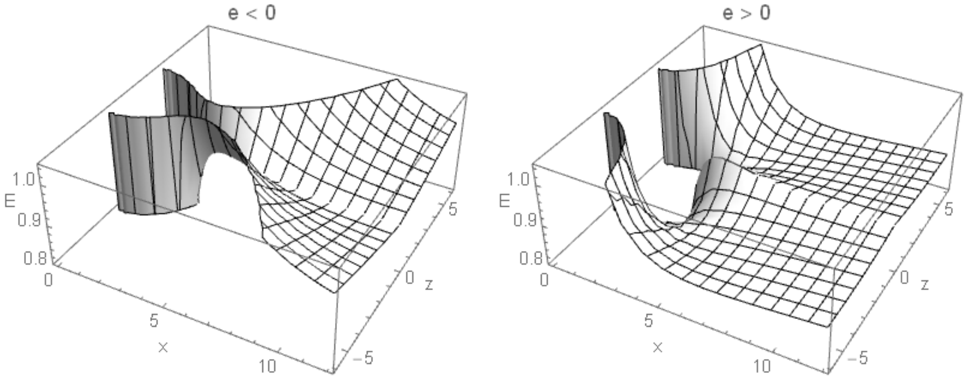

Figure 2. Effective potential $V_{\rm eff}(x,z)$ for charged particle motion around BH with split monopole magnetosphere.

covariant component of the electromagnetic four-vector potential A^μ has only one non-zero component A_ϕ

$$A_\mu = (0,0,0,\epsilon\,|\cos\theta|), \tag{3}$$

Here the parameter ϵ specifies the magnetic field intensity. Split monopole magnetic field has same symmetries as the Schwarzschild BH background (1) - is static and spherically symmetric.

Hereafter, we put $M = 1$, i.e., we use dimensionless radial coordinate r (and time coordinate t). Cartesian coordinates can be found by the coordinate transformations

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta). \tag{4}$$

The equations of motion for charged particle can be obtained using Hamiltonian formalism

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\zeta} = \frac{\partial H}{\partial \pi_\mu}, \quad \frac{\mathrm{d}\pi_\mu}{\mathrm{d}\zeta} = -\frac{\partial H}{\partial x^\mu}, \qquad H = \frac{1}{2}g^{\alpha\beta}(\pi_\alpha - qA_\alpha)(\pi_\beta - qA_\beta) + \frac{m^2}{2} = 0, \tag{5}$$

where the kinematical four-momentum $p^\mu = mu^\mu = \mathrm{d}x^\mu/\mathrm{d}\zeta$ is related to the generalized (canonical) four-momentum π^μ by the relation $\pi^\mu = p^\mu + qA^\mu$. The affine parameter ζ of the particle is related to its proper time τ by the relation $\zeta = \tau/m$.

Due to the symmetries of the Schwarzschild spacetime (1) and the magnetic field (3), one can easily find the conserved quantities that are particle energy and axial angular momentum

$$\mathcal{E} = \frac{E}{m} = -\frac{\pi_t}{m} = -g_{tt}u^t, \quad \mathcal{L} = \frac{L}{m} = \frac{\pi_\phi}{m} = g_{\phi\phi}u^\phi + \frac{q}{m}A_\phi. \tag{6}$$

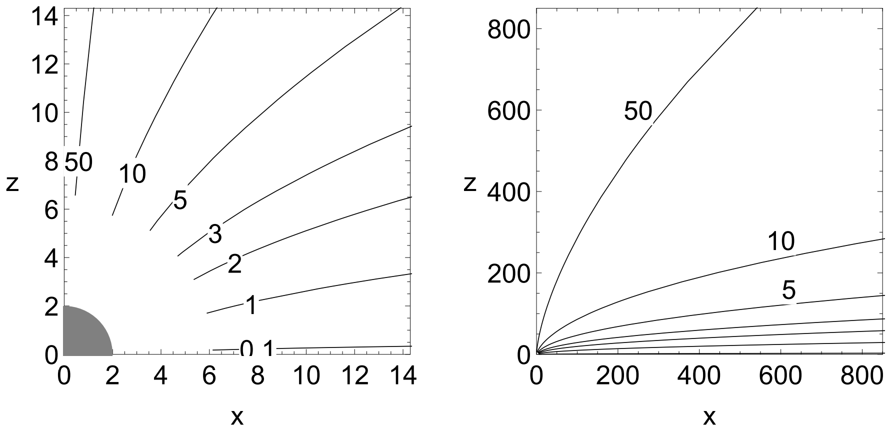

Figure 3. Positions of off equatorial stable circular orbits for various values of magnetic parameter e. The BH horizon is given by gray disk, while the points on each line denotes positions of stable circular orbits for different angular momenta L.

Using such symmetries one can rewrite the Hamiltonian (5) in the form

$$H = \frac{1}{2} g^{rr} p_r^2 + \frac{1}{2} g^{\theta\theta} p_\theta^2 + \frac{1}{2} g^{tt} E^2 + \frac{1}{2} g^{\phi\phi} (L - q A_\phi)^2 + \frac{1}{2} m^2 = H_{\mathrm{D}} + H_{\mathrm{P}}, \tag{7}$$

where we separated total Hamiltonian H into dynamical H_{D} (first two terms) and potential H_{P} (last two terms) parts.

For the description of charged particle motion we will use parameters: particle specific charge $\tilde{q}$, and magnetic field parameter e

$$\tilde{q} = q/m, \quad e = \epsilon \tilde{q}. \tag{8}$$

Energetic boundary for particle motion can be expressed from the equation (7)

$$\mathcal{E}^2 = V_{\mathrm{eff}}(r, \theta) \quad (\text{for } p_r = p_\theta = 0). \tag{9}$$

We introduced effective potential for charged particle $V_{\mathrm{eff}}(r, \theta)$ by the relation

$$V_{\mathrm{eff}}(r, \theta) \equiv -g_{tt} \left[g^{\phi\phi} \left(\mathcal{L} - \tilde{q} A_\phi \right)^2 + 1 \right] = \left(1 - \frac{2}{r} \right) \left[\frac{(\mathcal{L} - e|\cos\theta|)^2}{r^2 \cos^2\theta} + 1 \right]. \tag{10}$$

The effective potential $V_{\mathrm{eff}}(r, \theta)$ combines the influence of gravitational potential (first term) with the influence of central force potential given by the specific angular momentum $\mathcal{L}$ and electromagnetic potential energy (terms in square brackets). The positive angular momentum of a particle $\mathcal{L} > 0$ means that the particle is revolved in the counter-clockwise motion around the black hole in x-y plane. Example of effective potential $V_{\mathrm{eff}}(r, \theta)$ behavior can be found in Fig. 2. For charged particle we distinguish two following situations: *minus configuration* where $e < 0$ and *plus configuration* where $e > 0$.

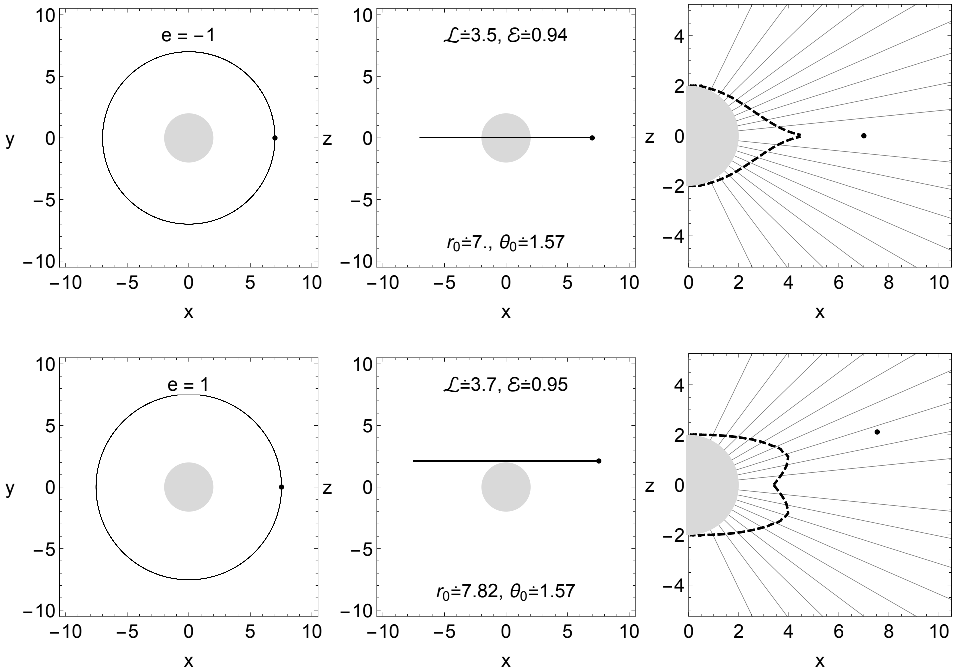

Figure 4. Example of stable circular equatorial and off-equatorial orbits around Schwarzschild BH (gray disk) with split monopole magnetosphere (gray lines). Particle trajectories are given by black curves, energetic boundary for the motion as dashed curves. Particle initially conditions are given in the figure, we also use $u^r_{(0)} = u^\theta_{(0)} = 0$.

3 TRAJECTORIES

Particles on circular orbits around compact object can form Keplerian accretion disk, with its inner edge given by innermost circular orbit (ISCO). The circular orbit parameters and ISCO position can be determined by examination of effective potential $V_{\text{eff}}(r,\theta)$ function. The stationary points of the effective potential $V_{\text{eff}}(r,\theta)$ function are given by

$$\partial_r V_{\text{eff}}(r,\theta) = 0, \quad \partial_\theta V_{\text{eff}}(r,\theta) = 0. \tag{11}$$

For minus configuration $e < 0$ the second equation in (11) has only one root at $\theta = \pi/2$. In another word, there is extrema of the $V_{\text{eff}}(r,\theta)$ function located in the equatorial plane only.

For plus configuration $e > 0$ there is the second equation in the extrema condition (11) has more solutions - off-equatorial plane minima of effective potential V_{eff} can exist. The positions of the off-equatorial plane minima, where stable off-equatorial plane circular or-

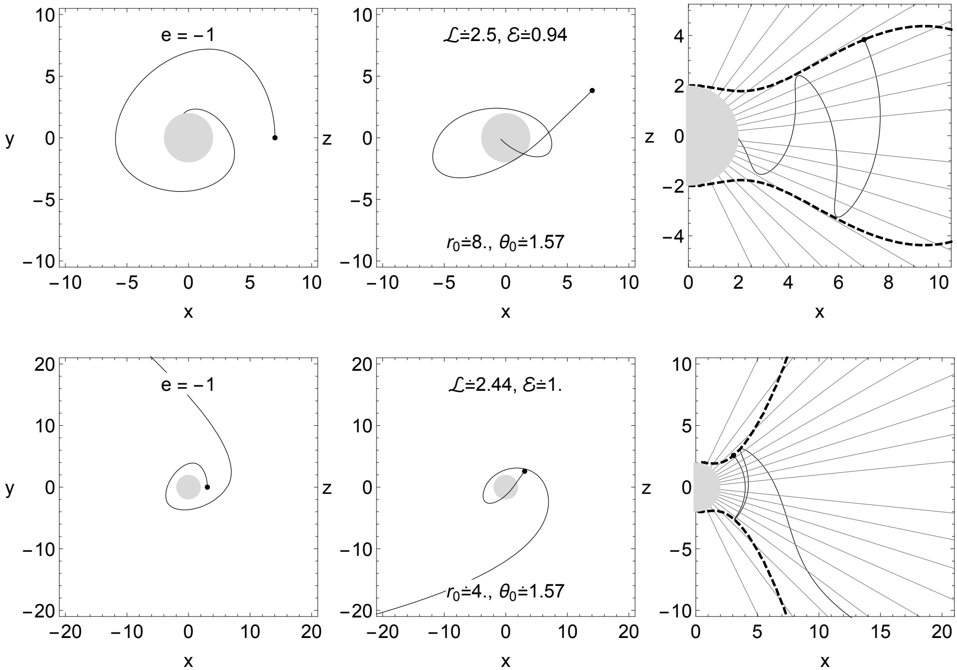

Figure 5. Charged particle capture and escape form Schwarzschild BH with split monopole magnetosphere, see Fig. 4 for description.

bits are located, are given by

$$r_{\rm min} = \frac{1}{2}\left(\mathcal{L}^2 - e^2 + \sqrt{12e^2 + e^4 - 12\mathcal{L}^2 - 2e^2\mathcal{L}^2 + \mathcal{L}^4}\right), \tag{12}$$

$$\theta_{\rm min} = \arctan\left(\frac{\sqrt{\mathcal{L}^2 - e^2}}{e}\right). \tag{13}$$

The closest orbit to the BH horizon - the innermost off-equatorial plane stable circular orbit $r_{\rm off\,ISCO}$ is located at $r_{\rm off\,ISCO} = 6M$, see Fig.3.

Examples of charged particle circular orbits in and off-equatorial plane can be found in Fig. 4. We also give trajectories of escaping particle and particle captured by BH in Fig. 5. Magnetic monopole field has the same spherical symmetry as Schwarzschild BH metric background and hence there are no chaotic trajectories.

4 FREQUENCIES

If charged test particle is slightly displaced from the equilibrium position, which is located in a minimum of the effective potential $V_{\rm eff}(r,\theta)$ at r_0 and θ_0, the particle will start to

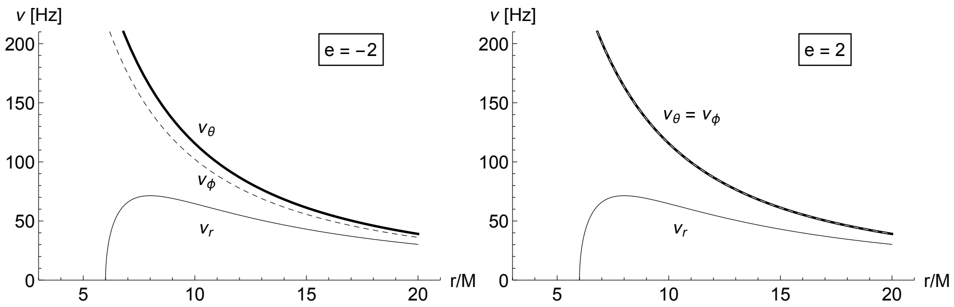

Figure 6. Frequencies for perturbation of in and off-equatorial plane circular orbits are ploted as function of radial position r. BH mass is take to be 10 solar masses.

oscillate around the minimum realizing thus epicyclic motion governed by linear harmonic oscillations. For harmonic oscillations around the minima of the effective potential $V_{\rm eff}$, the evolution of the displacement coordinates $r = r_0 + \delta r, \theta = \theta_0 + \delta\theta$ is governed by the equations

$$\ddot{\delta r} + \omega_{\rm r}^2\, \delta r = 0, \quad \ddot{\delta\theta} + \omega_\theta^2\, \delta\theta = 0, \tag{14}$$

where dot denotes derivative with respect to the proper time τ of the particle ($\dot{x} = \mathrm{d}x/\mathrm{d}\tau$). Locally measured angular frequencies of the harmonic oscillatory motion are given by

$$\omega_{\rm r}^2 = \frac{1}{g_{rr}}\frac{\partial^2 H_{\rm P}}{\partial r^2}, \quad \omega_\theta^2 = \frac{1}{g_{\theta\theta}}\frac{\partial^2 H_{\rm P}}{\partial\theta^2}, \quad \omega_\phi = \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \mathcal{L} g^{\phi\phi} + e g_{tt}, \tag{15}$$

where we added also the Keplerian (axial) frequency ω_ϕ. Frequencies for perturbations of circular orbit in equatorial plane ($e < 0$) are

$$\omega_{\rm r}^2 = \frac{r-6}{r^4}, \quad \omega_\theta^2 = \frac{e^2(r-3)+r^2}{r^5}, \quad \omega_\phi^2 = \frac{1}{r^3}, \tag{16}$$

while for perturbations off-equatorial plane ($e > 0$) we have

$$\omega_{\rm r}^2 = \frac{r-6}{r^4}, \quad \omega_\theta^2 = \frac{e^2(r-3)+r^2}{r^5}, \quad \omega_\phi^2 = \frac{e^2(r-3)+r^2}{r^5}. \tag{17}$$

The locally measured angular frequencies $\omega_{\rm r}, \omega_\theta, \omega_\phi$, given by $\omega_\beta = \mathrm{d}\beta/\mathrm{d}\tau$ where $\beta \in \{\mathrm{r}, \theta, \phi\}$, are connected to the angular frequencies measured by the static distant observers (in the physical units) by the gravitational redshift transformation

$$\nu_\beta = \frac{1}{2\pi}\frac{c^3}{GM}\frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{1}{2\pi}\frac{c^3}{GM}\frac{\omega_\beta}{-g^{tt}\mathcal{E}(r)}. \tag{18}$$

Behavior of the frequencies $\nu_{\rm r}(r), \nu_\theta(r)$ and $\nu_\phi(r)$, as functions of the radial coordinate r, are demonstrated in Fig. 6 for both positive and negative magnetic parameters. The charged particle oscillations with frequencies $\nu_{\rm r}(r), \nu_\theta(r)$ and $\nu_\phi(r)$, could be used for fitting of still unsolved quasi-periodic oscillations (QPOs) observed in many Galactic Low Mass X-Ray Binaries (Kološ et al., 2015, 2017).

5 CONCLUSIONS

As general relativistic magnetohydrodynamics simulations are showing, the real magnetic field around BH can have quite complicated character Komissarov (2004, 2005). In this work we used split monopole magnetic field as simple model for large scale BH magnetosphere with radial character.

Split monopole magnetosphere around Schwarzschild BH will influence charged particle motion: for negative magnetic field parameter $e < 0$ we have standard stable circular orbits located in equatorial plane, while for positive magnetic field parameter $e > 0$ we have stable circular orbits located off-equatorial only. Innermost stable circular orbit is for both cases at radius $r_{\mathrm{ISCO}} = r_{\mathrm{off\,ISCO}} = 6$ M. Charged particle fundamental frequencies can be significantly influenced by presence of split monopole magnetic field.

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