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Einstein–scalar field–square root nonlinear electrodynamics solution

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ABSTRACT
We study the influence of scalar fields on a specific model of nonlinear electrodynamics (the square root Lagrangian) spacetime. We show that the singular horizon created by scalar field in spherically symmetric static scalar-vacuum spacetimes is still present when nonlinear electrodynamics is added. For the obtained solution, we investigate the timelike geodesic motions of a test particle by studying the effective potential.

Keywords: Exact solution – black hole – scalar field – nonlinear electrodynamics – geodesic equation

1 INTRODUCTION

Studying scalar field when coupled to gravity whether minimally or nonminimally is an old subject in general relativity. The first exact spacetime solution with scalar field minimally coupled to gravity in this context was found by Fisher in 1948 (Fisher, 1948) and later was rediscovered several times (Wyman, 1981; Buchdahl, 1959; Bergmann and Leipnik, 1957; Janis et al., 1968). This scalar-vacuum static spherically symmetric solution was generalized to Einstein Maxwell scalar field solution (Penney, 1969; Janis et al., 1969; Uhlíř and Dittrich, 1973; Teixeira et al., 1974, 1976; Eriş and Gürses, 1977; Banerjee and Choudhury, 1977). The most famous and frequently used form of Fisher solution is the one described in (Janis et al., 1968), where they showed that such spacetime contains a singular pointlike event horizon. This solution is referred to as Janis–Newuman–Winicour (JNW) spacetime.

Presence of naked singularities or irregular horizons was shown to be typical for scalar field spacetimes by J. E. Chase in 1970 in what is now known as the "Chase theorem" (Chase, 1970). According to it, roughly any static spherically symmetric vacuum solution minimally coupled to massless scalar field can not have a regular horizon, any potential horizon is necessarily the locus of a curvature singularity (see (Tafel, 2014) for generalization including potential for the scalar field). These results are connected to scalar no-hair

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theorem nicely reviewed in (Herdeiro and Radu, 2015) where they study four dimensional asymptotically flat black holes with scalar hair in various types of scalar field models coupled to gravity.

Our motivation is to confirm whether the Chase theorem still holds when, additionally to massless scalar field, other sources are present, such as Nonlinear Electrodynamics (NE). Nonlinear electrodynamics is a nonlinear theory of electromagnetic field and various models exist with different Lagrangians. The most famous and successful model is Born–Infeld (Born and Infeld, 1934) which in the weak field limit goes to the linear Maxwell theory and in the strong field limit its Lagrangian tends to $\sim \sqrt{F_{\mu\nu}F^{\mu\nu}}$ (square root model), with $F_{\mu\nu}$ being electromagnetic tensor.

Since we were not able to find an exact solution for Born–Infeld model when scalar field minimally coupled to gravity is present, we chose the "square root" model as its approximation in the strong field regime. We believe one can extend any results related to an event horizon in the square root model to Born–Infeld model since horizons appear in strong field regime. Apart from this reason, square root Lagrangians were studied because of their interesting properties long time ago (Nielsen and Olesen, 1973; Gaete and Guendelman, 2006; Vasihoun and Guendelman, 2014) even before the rise in popularity of NE where it gained attention recently.

Previously, we studied square root model NE in Kundt class of geometries which contain exact gravitational waves (Tahamtan and Svitek, 2017).

In (Svítek et al., 2020; Tahamtan and Svitek, 2014), it is shown that the spacetime singularity sourced by static spherically symmetric scalar field is resolved at the quantum level. In (Svitek and Tahamtan, 2016), we show that scalar-field sources in static, highly symmetric geometries (JNW) tend to vanish in the ultraboost limit instead of being converted into waves.

Scalar field solutions can be generalized beyond spherical symmetry to truly dynamical situation (Tahamtan and Svitek, 2015, 2016) using Robinson–Trautman class of geometries. The results confirm no-hair theorem in the asymptotic stationary limit. This class of geometries can be coupled to NE as well (Tahamtan and Svitek, 2016).

2 SCALAR FIELD AND SQUARE ROOT LAGRANGIAN

We consider the following action, describing a scalar field and an electromagnetic field in the form of nonlinear electrodynamics minimally coupled to gravity,

$$S = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\left[\mathcal{R} + \nabla_\mu\varphi\nabla^\mu\varphi + \mathcal{L}(F)\right], \tag{1}$$

where $\mathcal{R}$ is the Ricci scalar for the metric $g_{\mu\nu}$ (we use units convention $c = \hbar = 8\pi G = 1$). The massless scalar field φ is considered real and the NE Lagrangian $\mathcal{L}(F)$ is assumed to be an arbitrary function of the electromagnetic field invariant $F = F_{\mu\nu}F^{\mu\nu}$ constructed from a closed Maxwell 2-form $F_{\mu\nu}$.

We consider the static spherically symmetric metric

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + R(r)^2\mathrm{d}\Omega^2\,, \tag{2}$$

where $d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2$. We assume t, r, θ, ϕ coordinate ordering.

By applying the variation with respect to the metric using the action (1), we obtain Einstein equations

$$G^\mu{}_\nu = T^\mu{}_\nu = {}^{\mathrm{SF}}T^\mu{}_\nu + {}^{\mathrm{EM}}T^\mu{}_\nu. \tag{3}$$

where the superscript SF indicates scalar field and EM electromagnetic contribution to energy momentum tensor. For finding an exact solution for our metric functions corresponding to the scalar field and nonlinear electrodynamics sources, we first express the energy momentum tensors for these sources explicitly.

The energy momentum tensor generated by the scalar field is given by

$${}^{\mathrm{SF}}T_{\mu\nu} = \nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\, g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi \tag{4}$$

which for a radial scalar field and our metric anzats (2) reduces to

$${}^{\mathrm{SF}}T^\mu{}_\nu = \frac{f\,\varphi_{,r}^2}{2} diag\{-1, 1, -1, -1\}\ . \tag{5}$$

The wave equation of a massless scalar field ($\Box\varphi = 0$, where $\Box$ is a standard d'Alembert operator) with respect to our metric (2) leads to

$$f\,\varphi_{,r}\, R^2 = const. \tag{6}$$

And the electromagnetic energy momentum tensor contribution is defined as following

$${}^{\mathrm{EM}}T^\mu{}_\nu = \frac{1}{2}\left\{\delta^\mu{}_\nu\mathcal{L} - (F_{\nu\lambda}F^{\mu\lambda})\mathcal{L}_F\right\}, \tag{7}$$

in which $\mathcal{L}_F = \frac{d\mathcal{L}(F)}{dF}$. Obviously for the Maxwell case $\mathcal{L} = -F$ and $\mathcal{L}_F = -1$.

For our particular choice of nonlinear electrodynamics model, square root Lagrangian $\mathcal{L} = -\sqrt{F}$, the energy momentum tensor simplifies considerably

$${}^{\mathrm{NE}}T^\mu{}_\nu = diag\left\{-\frac{\sqrt{F}}{2}, -\frac{\sqrt{F}}{2}, 0, 0\right\}\ . \tag{8}$$

Since our spacetime is static and spherically symmetric, we assume this to hold for electromagnetic field as well and consider the following electromagnetic field two-form for purely magnetic field

$$\mathbf{F} = F_{\theta\phi}\, d\theta \wedge d\phi\,, \tag{9}$$

where $F_{\theta\phi} = q_m \sin\theta$ and q_m can be considered as a magnetic charge. All the modified Maxwell equations (the source–free nonlinear Maxwell equations are $d\boldsymbol{F} = 0, d(\mathcal{L}_F\, {}^*\boldsymbol{F}) = 0$, where ${}^*\boldsymbol{F}$ is a dual of electromagnetic two-form $\boldsymbol{F}$) are satisfied trivially. The electromagnetic invariant $F = F_{\mu\nu}F^{\mu\nu}$ becomes

$$F = \frac{2\,q_m^2}{R^4}\ . \tag{10}$$

We start to solve the coupled system by considering tt and rr components of Einstein equations (3), namely $G^t{}_t - G^r{}_r = T^t{}_t - T^r{}_r$ and we immediately obtain

$$\varphi_{,r}^2 = -\frac{2\,R_{,rr}}{R}. \tag{11}$$

From the above equation and (6) we are able to find f in terms of R

$$f = \sqrt{-\frac{C_0^2}{2\,R^3\,R_{,rr}}}\,. \tag{12}$$

The rest of Einstein equations will constrain the form of R. From $G^t{}_t - T^t{}_t = 0$, we get

$$f\left(\frac{R_{,r}}{R}\right)^2 + \frac{R_{,r}}{R}\,f_{,r} - \frac{1}{R^2} + f\,\frac{R_{,rr}}{R} + \frac{q_m}{\sqrt{2}}\frac{1}{R^2} = 0\,, \tag{13}$$

which together with (12) gives the following expressions for R, f and from (11) for the scalar field φ

$$R(r) = \sqrt{\beta^2\left(r+\tilde{C}_1\right)\left(r-\tilde{C}_2\right) - C_0^2} \times \exp\left(-\Omega(r)\right)\,, \tag{14}$$

$$f(r) = -\frac{e^{2\Omega(r)}}{\beta\,\sqrt{2}}\,, \tag{15}$$

$$\varphi(r) = \frac{2\sqrt{2}C_0}{\beta\left(\tilde{C}_1+\tilde{C}_2\right)}\,\Omega(r)\,, \tag{16}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are integration constants and we introduced parameters β, ρ and a function $\Omega(r)$ in the following way

$$\beta = \left(q_m - \sqrt{2}\right), \tag{17}$$

$$\rho = \sqrt{\beta^2\left(\tilde{C}_1+\tilde{C}_2\right)^2 + 4C_0^2}, \tag{18}$$

$$\Omega(r) = \frac{\beta\left(\tilde{C}_1+\tilde{C}_2\right)}{2\rho}\ln\left(\frac{r-r_0}{r-\tilde{r}_0}\right) \tag{19}$$

where $r_0 = \frac{1}{2}\left(\tilde{C}_2 - \tilde{C}_1 - \rho/\beta\right)$, $\tilde{r}_0 = r_0 + \rho/\beta$ and β should be negative for preserving the metric signature. After some simplifications, the equations (14) and (15) become

$$R(r) = \sqrt{\beta^2\,(r-r_0)\,(r-\tilde{r}_0)}\left[\frac{r-\tilde{r}_0}{r-r_0}\right]^{\frac{\nu}{2}}\,, \tag{20}$$

$$f(r) = -\frac{1}{\beta\,\sqrt{2}}\left[\frac{r-r_0}{r-\tilde{r}_0}\right]^{\nu}\,, \tag{21}$$

where $\nu = \frac{|\beta(\tilde{C_1}+\tilde{C_2})|}{\rho} \geq 0$.

It is clear that f is vanishing at $r = r_0$ indicating horizon. Behavior of R is driven by the power of $(r - r_0)$, which is $\frac{\nu-1}{2}$. Depending on whether $\nu \lesseqqgtr 1$, R would be zero, finite or diverge. Considering the definition for ρ from (18), it is clear that $\nu < 1$ if we have $C_0 \neq 0$ (nontrivial scalar field). Thus at $r = r_0$ the function R is vanishing and this location corresponds to a point instead of a sphere.

Note that since β is negative, $r_0 > \tilde{r}_0$. So at r_0 there is an outermost horizon and it is a candidate for the outer event horizon of a black hole but we need to see the behavior of Ricci scalar at $r = r_0$ to determine its regularity.

Ricci scalar with respect to our metric anzats (2) is

$$Ricci = -f_{,rr} - \frac{4}{R}\left(f R_{,r}\right)_{,r} - 2f\left(\frac{R_{,r}}{R}\right)^2 + \frac{2}{R^2} \tag{22}$$

and using (22), we obtain the following expression

$$Ricci \sim (r - r_0)^{\nu-2}\,.$$

Since $\nu < 1$ the Ricci scalar at $r = r_0$ is clearly diverging.

It is clear from (20) and (21) that it is difficult to obtain the metric function f in terms of R, for this reason we use a parametric plot for f in terms of R to see the behavior in terms of the areal radius which has better physical interpretation (see Fig. 1). The behavior is clearly monotonous and the curves for different β approach the location of curvature singularity at $R = 0$ smoothly.

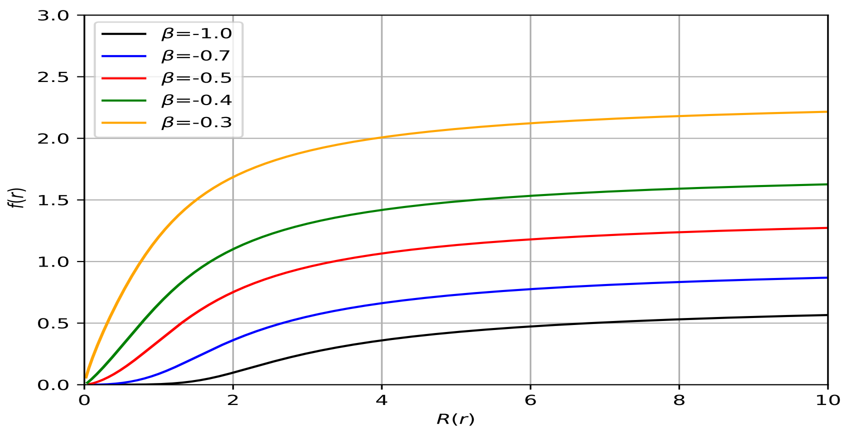

Figure 1. The metric function $f(r)$ in terms of $R(r)$ with different values of β.

So in our solution, the event horizon is also a true singularity which confirms the role of scalar field in spoiling horizon regularity even in this NE model. Since this is the stationary state of geometry it shows that the no-hair theorem is valid in this case as well since we have not found black hole spacetime with both nongravitational fields being nontrivial.

The scalar field (16) becomes

$$\varphi(r) = \frac{\sqrt{2}C_0}{\rho} \ln\left[\frac{r - r_0}{r - \tilde{r}_0}\right] \tag{23}$$

and it is clear that at $r = r_0$, it diverges as well and the same applies to electromagnetic invariant (10) and therefore to NE energy momentum tensor (8).

The obtained solution, (20) and (21), is a NE generalization of Janis, Newmann and Winicour solution (Janis et al., 1968) and the original solution is recovered for $q_m = 0$ while as well setting $\tilde{C}_1 = \tilde{C}_2$.

If we consider a special case when the scalar field vanishes, $C_0 = 0$, then necessarily $\nu = 1$ and the solution in (20) and (21) will be equivalent to (Tahamtan, 2020) upon trivial changes in coordinates and constants.

If we assume that both $\tilde{C}_1$ and $\tilde{C}_2$ vanish then the form of the metric functions simplifies

$$R(r) = \sqrt{\beta^2\, r^2 - C_0^2}\,, \tag{24}$$

$$f(r) = -\,\frac{1}{\beta\,\sqrt{2}}\,, \tag{25}$$

leading to spacetime containing timelike naked singularity. When q_m in β vanishes then the solution becomes equivalent to (Tahamtan and Svitek, 2016) with some trivial redefinition of coordinate r.

All the above mentioned solutions with nontrivial scalar field do not possess regular horizon. Although Maxwell theory and square root NE are significantly different since both their weak field limit and strong field behavior disagree, when coupled to scalar field they both produce singular horizon or naked singularity. This indicates dominant negative role of the scalar field in horizon formation. Note that there is crucial difference already for solutions without scalar field because square root model geometry (Tahamtan, 2020) only possesses single horizon compared to Reissner–Nordström solution which can have two and global asymptotics disagree as well. Nevertheless, the scalar field produces solutions with similar characteristic — singular horizons — in both cases.

3 GEODESIC MOTION

We study particle motion in order to understand the physical properties of the spacetime under consideration. We will use the variational principle and the Euler—Lagrange equations for timelike geodesics. The Lagrangian reduces to kinetic part only and has the following form

$$2\,L = -f\,\dot{t}^2 + \frac{\dot{r}^2}{f} + R^2\left(\dot{\theta}^2 + \sin\theta^2\,\dot{\phi}^2\right) \tag{26}$$

in which dot denotes the derivative with respect to the proper time τ. Because of spherical symmetry we study the particle motion in equatorial plane, so $\theta = \frac{\pi}{2}$. By using Euler–Lagrange equations we find two conserved quantities, E (energy) and l (orbital angular momentum), as expected for static and spherically symmetric spacetime which admits two Killing vectors $(\partial_t, \partial_\phi)$. The energy and angular momentum are given by

$$E = f\,\dot{t} \tag{27}$$

$$l = R^2\,\dot{\phi} \tag{28}$$

Substituting the above expressions into (26), we obtain equation for radial component of fourvelocity corresponding to timelike geodesic motion

$$\dot{r}^2 + V_{\text{eff}} = E^2 \tag{29}$$

where V_{eff} is the effective potential given by

$$V_{\text{eff}} = f\left(\frac{l^2}{R^2} + 1\right). \tag{30}$$

For plotting the effective potential, we consider the following values of constants: $C_0 = 1, \tilde{C_1} = -3, \tilde{C_2} = 1$. The only remaining constant parameter is β and we plot our graphs for its different values. Since the domain for β is $(-\infty, 0)$ the domain of $\nu = \frac{|\beta|}{\sqrt{\beta^2+1}}$ is $(0, 1)$. Because $\nu = 1$ is attained asymptotically for $\beta \to \infty$ more interesting changes in behavior happen for smaller β.

First, we plot the effective potential for zero angular momentum, $l = 0$, in this case the potential and the metric function f would be the same (30), see Fig. 2. When the absolute value of β is increasing the effective potential (metric function f) is decreasing. The zeros are the spacetime singularity points which appear at different r for different β but all correspond to $R(r) = 0$. This plot shows that the radially falling particle approaches singularity with velocity depending on the value of β.

Next, we plot the effective potential for $l = 1$ and different values of β (same as those used for $l = 0$ case), see Fig. 3. Here, similar to case when $l = 0$ the potential values are decreasing with increasing absolute value of β. As it is shown in the plot, for some values of β the effective potential character changes and one global minimum appears indicating stable circular orbits.

4 CONCLUSION AND FINAL REMARKS

We showed that static scalar field spacetime coupled to $\sim -\sqrt{F}$ Lagrangian which captures the strong field regime of many NE models (e.g., Born–Infeld) admits generalized solution of Janis–Newman–Winicour. Similar to all minimally coupled scalar field solutions, the spacetime has an irregular horizon which is in agreement with the Chase theorem. Our result and the previous ones show that the effect of scalar field on the spacetime geometry is dominant. Note that in the absence of scalar field, square root model Lagrangian solution represents a black hole solution with regular horizon.

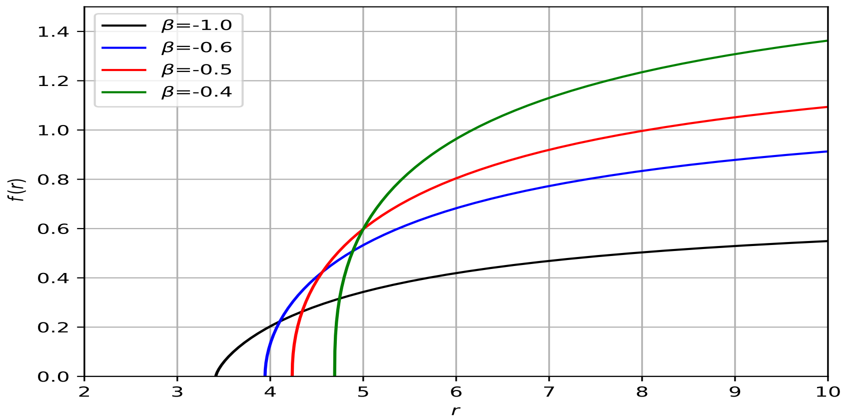

Figure 2. Effective potential V_{eff} for $l = 0$ and different values of β.

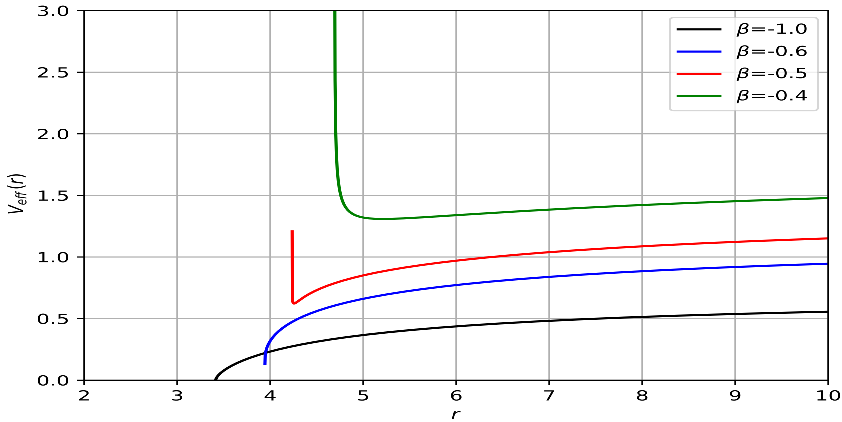

Figure 3. Effective potential V_{eff} for $l = 1$ and different values of β.

Furthermore, we studied timelike geodesic motion of a test particle. The obtained effective potential shows that for nonzero angular momentum and certain values of parameter β it is possible to have stable circular orbits. These stable circular orbits around singularity could give rise to disc configurations around the singularity and further study can give clear observational signatures such objects might exhibit.

In future, we will generalize this solution to massive scalar field with potentials.

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