

Tidal deformability of ultracompact Schwarzschild stars and their approach to the black hole limit

Camilo Posada

Institute of Physics in Opava, Silesian University in Opava,
Bezručovo nám. 13, CZ-746 01 Opava, Czech Republic
camilo.posada@physics.slu.cz

ABSTRACT

A well known result in general relativity, is that the tidal Love numbers of black holes vanish. In contrast, different configurations to a black hole may have non-vanishing Love numbers. For instance, it has been conjectured recently that the Love number of generic exotic compact objects (ECOs) shows a logarithmic behaviour. Here we analyse the ultracompact Schwarzschild star which allows the compactness to cross and go beyond the Buchdahl limit. This Schwarzschild star has been shown to be a good black hole mimicker, moreover it has been found that the Love number of these objects approaches to zero as their compactness approaches the black hole limit. Here we complement those results, by showing that the Love number for these configurations follows an exponentially decaying behaviour rather than the logarithmic behaviour proposed for generic ECOs.

Keywords: Tidal deformability – interior solutions – black hole mimicker – gravastar

1 INTRODUCTION

Black holes (BHs) are one of the most intriguing predictions of Einstein’s classical general relativity (GR). Beyond their elegant mathematical structure (Chandrasekhar, 1985), they are also the main candidates to explain most of the astrophysical observations (Celotti et al., 1999). Nevertheless, besides their curvature singularity in the interior geometry, which is believed will be ‘removed’ by a consistent quantum theory of gravity, BHs present certain paradoxes which remain puzzling. For instance, the interior geometry of mathematical BHs shows some unphysical behaviour, such as the closed timelike geodesics in the Kerr spacetime (Hawking and Ellis, 1973). On the other hand, the event horizon is at the root of the so-called ‘information paradox’, which remains one of the central problems in black hole physics (Wald, 2001).

As a consequence of the BH paradoxes, a number of models of ‘regular BHs’ or ‘BH mimickers’ (also known as exotic compact objects (ECOs)) have been proposed in the

literature (see, e.g. (Cardoso and Pani, 2019) and references therein). Regular BHs are constructed with different non-singular interiors, but in such a way that they reproduce the well-known geometries of BH solutions in GR. For instance, the gravitational condensate star, or gravastar (Mazur and Mottola, 2001, 2004), is composed of a non-singular de Sitter interior with negative pressure $p = -\epsilon$, but with positive vacuum energy $\epsilon > 0$, which is matched to the exterior Schwarzschild solution with $p = \epsilon = 0$. The ‘gluing’ of both geometries was done by introducing an infinitesimal thin shell of ultra-stiff matter.

In connection with gravastars, Mazur and Mottola (2015) revisited the well-known Schwarzschild’s interior solution with uniform density, or *Schwarzschild star* (see e.g. (Glendenning, 2000)). The Schwarzschild star manifests a divergence in the central pressure when its compactness reaches the Buchdahl bound $M/R = 4/9$. The importance of this limit relies on the fact that it is independent of the equation of state (EOS) of the configuration, as shown by (Buchdahl, 1959), under the assumption of isotropic pressure, positive energy density and monotonically decreasing with the distance r . Thus, the Schwarzschild star represents a toy model which saturates the Buchdahl bound and should be considered as the limiting case of an ultra-stiff EOS; incidentally, configurations with compactness higher than $4/9$ have been usually assumed as unphysical.

Nevertheless, some interesting features become apparent when one considers the Schwarzschild star beyond the Buchdahl limit, $R_S < R < (9/8)R_S$, where $R_S \equiv 2M$. First of all, the pole where the pressure is divergent moves out from the origin up to a surface of radius $R_0 = 3R\sqrt{1 - (4/9)(R/M)} < R$, and a regular interior region with negative pressure emerges naturally in the regime $0 < r < R_0$; meanwhile the pressure remains positive in the region $R_0 < r < R$. In the limit when $R_0 \rightarrow R_S^-$, from below, and $R \rightarrow R_S^+$, from above, the ultracompact Schwarzschild star becomes essentially the gravastar proposed by Mazur and Mottola (2015). It is important to remark that the Schwarzschild star *evades* the Buchdahl limit by having an anisotropic stress at the surface R_0 . In the limiting case when $R_0 = R = R_S$, the interior static de Sitter is matched to the exterior Schwarzschild geometry through a boundary layer located at their respective horizons $R_S = H$, where H is related to the de Sitter energy density by $\epsilon = 3H^2/(8\pi)$. Furthermore, there is a discontinuity $[\kappa]$ in the surface gravities, which produces a surface tension $\tau_s = 1/(8\pi R_S)$ and incidentally a δ -function stress tensor which replaces the BH horizon.

Some of the physical properties and observational signatures of the ultracompact Schwarzschild stars have been studied recently in the literature. For instance, a time-dependent model was proposed by Beltracchi and Gondolo (2019). On the other hand, these configurations seem to be stable against radial oscillations (Posada and Chirenti, 2019). The analysis of axial modes was carried out in (Konoplya et al., 2019), where it was found that the Schwarzschild stars are stable against axial perturbations. Moreover, the Schwarzschild star can ‘mimic’ very well the gravitational wave response of a BH at $l = 2$ and higher multipoles because it approaches the Schwarzschild BH spectrum as closely as possible. This is due to the fact that the null surface R_0 provides the same boundary conditions for the quasi-normal modes as for the case of a BH. An extension to the anisotropic case using the minimal geometric deformation (MGD) was developed by Ovalle et al. (2019).

An early model for a slowly rotating ultracompact Schwarzschild star was proposed by Posada (2017) using the Hartle-Thorne framework (Hartle, 1967; Hartle and Thorne, 1968).

However, the results reported there are marred by a wrong assumption in the regime beyond the Buchdahl limit after making a coordinate transformation. This proposal has been surpassed recently by Beltracchi et al. (2021), where they developed a model for a slowly rotating gravastar to second order in the rotation. These authors found that the exterior metric to a slowly rotating gravastar is precisely that of the Kerr spacetime. Therefore, it's not possible to tell a gravastar from a Kerr BH by any observation, such as accretion disk processes or light ring images.

An alternative for, potentially, distinguishing ECOs from BHS is through their tidal deformability. A compelling result in GR is that the tidal Love numbers of Schwarzschild BHs are zero (Damour and Nagar, 2009; Binington and Poisson, 2009; Hui et al., 2021; Chia, 2021; Charalambous et al., 2021; Poisson, 2021). On the other hand, it has been found that the Love numbers of general ECOs scale as $k \sim 1/\log \xi$, where ξ is a parameter which measures how much the object deviates from the BH geometry (Cardoso et al., 2017). Recently Chirenti et al. (2020) studied the tidal Love number of ultracompact Schwarzschild stars, below and beyond the Buchdahl limit. These authors found that the Love number of these configurations tends to zero as the compactness approaches the BH limit. Thus, they concluded that the vanishing of the Love number is not a unique property of BHS. Instead, it's a consequence of the approach to the Schwarzschild limit.

In this paper, we will review the main results of the tidal deformability of ultracompact Schwarzschild stars presented by Chirenti et al. (2020). As an addition to those results, we will show that the tidal Love number k_2 for ultracompact Schwarzschild stars does not follow the $1/\log \xi$ proposed by (Cardoso et al., 2017). Instead, we found that k_2 decays exponentially as a function of the compactness.

2 TIDAL DEFORMABILITY

2.1 General formulation

The tidal Love number quantifies the deformations of the quadrupole moments of a star induced by external fields, which are connected through the relation (Hinderer, 2008; Damour and Nagar, 2009)

$$Q_{ij} = -\frac{2k_2 R^5}{3} E_{ij} \equiv -\Lambda E_{ij}. \quad (1)$$

where k_2 is the Love number and Λ is the tidal deformability. It is conventional to introduce the dimensionless tidal deformability

$$\bar{\Lambda} = \Lambda/M^5 = 2k_2/(3C^5), \quad (2)$$

here $C \equiv M/R$ denotes the compactness of the configuration. Following Damour and Nagar (2009), the unperturbed spacetime of a nonrotating star is described, generally, by the standard metric

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2, \quad (3)$$

and the even-parity metric perturbation $H = H_0 = H_2$ is governed by the following equation

$$\frac{d^2 H}{dr^2} + C_1(r) \frac{dH}{dr} + C_0(r)H = 0, \quad (4)$$

where the coefficients C_1 and C_0 are given by

$$C_1(r) = \frac{2}{r} + e^\lambda \left[\frac{2m}{r^2} + 4\pi r(p - \epsilon) \right], \quad (5)$$

$$C_0(r) = e^\lambda \left[-\frac{l(l+1)}{r^2} + 4\pi(\epsilon + p) \frac{d\epsilon}{dp} + 4\pi(5\epsilon + 9p) \right] - \left(\frac{dv}{dr} \right)^2. \quad (6)$$

In order to simplify the form of the perturbation equation (4), it is conventional to introduce the logarithmic derivative $h(r) \equiv (r/H)dH/dr$. Substituting $h(r)$ into Eq. (4), we obtain a Riccati-type equation in the form Damour and Nagar (2009)

$$r \frac{dh}{dr} + h(h-1) + rC_1 h + r^2 C_0 = 0, \quad (7)$$

with the regular solution near the origin, $h(r) \simeq l$. Finally, the Love number k_2 can be determined using the following expression

$$\begin{aligned} k_2(C, h_R) = & \frac{8}{5} (1 - 2C)^2 C^5 [2C(h_R - 1) - h_R + 2] \times \{2C[4(h_R + 1)C^4 \\ & + (6h_R - 4)C^3 + (26 - 22h_R)C^2 + 3(5h_R - 8)C - 3h_R + 6] \\ & + 3(1 - 2C)^2 [2C(h_R - 1) - h_R + 2] \log(1 - 2C)\}^{-1}. \end{aligned} \quad (8)$$

where h_R is the value of h at the surface $r = R$.

2.2 Schwarzschild stars

In this section, we discuss the homogeneous configurations with constant density. Although uniform-density stars are only an approximation for a realistic compact object, they are useful toy models which are described by Schwarzschild's interior solution (Glendenning, 2000). Moreover, there are various reasons to consider them in detail (see, e.g. Harrison et al. (1965)).

The tidal Love number for constant-density stars, or Schwarzschild stars, was studied by (Damour and Nagar, 2009; Postnikov et al., 2010; Chan et al., 2015) and more recently by (Chirenti et al., 2020). The details of the computation were discussed in these papers, so we refer the reader to those works.

The interior Schwarzschild solution describes a configuration of constant energy density ϵ . For convenience, it can be written in terms of the auxiliary function y (Chandrasekhar and Miller, 1974; Posada and Chirenti, 2019; Chirenti et al., 2020) defined as

$$y^2 = 1 - \left(\frac{r}{\alpha} \right)^2, \quad \text{with} \quad \alpha^2 = \frac{3}{8\pi\epsilon} \equiv \frac{R^3}{R_S}, \quad (9)$$

where R is the radius of the star and $R_S \equiv 2M$ is the Schwarzschild radius. The function y is defined in the range $[1, y_1]$ where $y_1 \equiv y(R)$ is the corresponding value at the surface. The interior metric functions are given by

$$e^\nu = \frac{1}{4}(3y_1 - y)^2 \quad \text{and} \quad e^\lambda = \frac{1}{y^2}, \quad (10)$$

which match continuously, at the surface $r = R$, with the exterior Schwarzschild solution. The pressure p in the interior is found to be

$$p = \epsilon \left(\frac{y - y_1}{3y_1 - y} \right), \quad (11)$$

which vanishes at the surface $r = R$. The central pressure diverges when the compactness reaches the Buchdahl bound $M/R = 4/9$ (Buchdahl, 1959).

It is convenient to introduce a new coordinate x given by (Chandrasekhar and Miller, 1974)

$$x \equiv 1 - y = 1 - \sqrt{1 - \left(\frac{r}{\alpha}\right)^2}, \quad (12)$$

which is defined in the range $[0, x_1]$ where $x_1 = 1 - y_1 \equiv x(R)$, which depends on the compactness. The interior metric functions (10) take the form

$$e^{\nu(x)} = \frac{1}{4}(k + x)^2, \quad \text{and} \quad e^{\lambda(x)} = \frac{1}{(1 - x)^2}, \quad (13)$$

and the pressure (11) reads now

$$p = \epsilon \left(\frac{1 - x - y_1}{\kappa + x} \right), \quad (14)$$

where the constant κ is defined as

$$\kappa \equiv 3y_1 - 1. \quad (15)$$

Note that $\kappa > 0$ when $C < 4/9$, and, $-1 < \kappa \leq 0$ in the regime beyond Buchdahl. Note that Eq. (14), has a regular singular point at $x_0 \equiv -\kappa$. The careful analysis of this singularity for the computation of the Love number was done in (Chirenti et al., 2020), so we refer the reader to that paper for more details.

The same singular point appears when one considers the extension of the ultracompact Schwarzschild star to slow rotation, using the equations derived by Hartle (Chandrasekhar and Miller, 1974; Posada, 2017). However, the results presented by Posada (2017) are marred after assuming, wrongly, an absolute value of κ when the compactness goes beyond the Buchdahl limit ¹.

¹ The author acknowledges Emil Mottola and Philip Beltracchi for calling his attention to this point.

3 TIDAL DEFORMABILITY OF ULTRACOMPACT SCHWARZSCHILD STARS

In this section, we summarise the results of the tidal deformability of ultracompact Schwarzschild stars reported by Chirenti et al. (2020). We used the transformation (12) in the perturbation equation (7), which facilitates the computation when the compactness goes beyond the Buchdahl bound. Profiles of the solutions to the perturbation equation for certain values of the compactness are presented in Figs. 1 and 2 in (Chirenti et al., 2020). Of particular interest is the value of k_2 at the Buchdahl limit, which was found to be $\kappa_2^{\text{Buch}} = 0.0017103$, which is in excellent agreement with the result reported by Damour and Nagar (2009).

The value of k_2 is computed in the following way; the structure equations (13)–(14) are substituted into the coefficients (5) and (6), which consequently determine Eq. (7). Given the condition at the origin $h(r) = l$, one solves numerically Eq. (7) for $h(r)^-$, in the interior of the star, from the center (or, rather, some cutoff value, $r_0 = 10^{-7}$), outwards up to the surface $r = R$. Some care must be taken here regarding the constant density condition. Note that Eq. (6) contains a term in the form $(\epsilon + p)(d\epsilon/dp)$ which vanishes in the limit $\epsilon = \text{constant}$. Given the discontinuity of the density at the surface, this term contributes a δ -function which must be taken into account to obtain h_R^+ (see the discussion in (Damour and Nagar, 2009; Postnikov et al., 2010)). This correction is given by

$$h_R^+ = h_R^- - \left(\frac{4\pi R^3 \epsilon}{M} \right)^- \quad (16)$$

which gives,

$$h_R^+ = h_R^- - 3 \quad (17)$$

In Fig. 1, we show the profile of the tidal Love number k_2 , as a function of the compactness M/R , for Schwarzschild stars below and beyond the Buchdahl limit (Chirenti et al., 2020). The inset shows a zoom of the region near the Buchdahl limit up to the Schwarzschild radius. The results of k_2 for $C \leq 4/9$ are in very good agreement with those reported by Damour and Nagar (2009). Note how k_2 approaches continuously to zero as the compactness approaches the Schwarzschild limit. Stars with compactness bigger than $4/9$ show a region of negative pressure in the interval $x \in [0, x_0]$ with $x_0 = -\kappa$. In the limit when $C \rightarrow 1/2$ from above, and $x_0 \rightarrow x_1$ from below, the central region of negative pressure covers the whole interior of the star. Thus, in this limit, the ultracompact Schwarzschild star becomes essentially the gravastar (Mazur and Mottola, 2015).

Note the striking ‘quenching’, in three orders of magnitude, of the tidal Love number, for ultracompact Schwarzschild stars, from 0.75, for $C \rightarrow 0$, to 0.0017103 at the Buchdahl bound, and the subsequent approach to zero as the compactness approaches $1/2$. This rapid quenching of k_2 clearly indicates that these configurations do not follow the logarithmic behaviour suggested by Cardoso et al. (2017) as a ‘generic feature’ of ECOs with an exterior geometry arbitrarily close to the BH geometry. To see this, in Fig. 2, we show the same

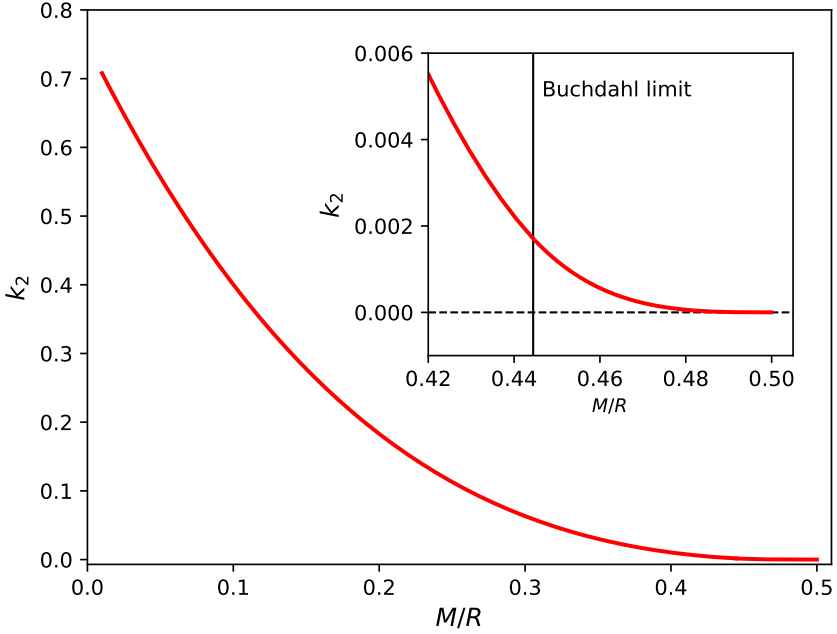


Figure 1. Tidal Love number k_2 , as a function of the compactness, for Schwarzschild stars below and beyond the Buchdahl limit. Note how k_2 approaches smoothly and continuously to zero, as the compactness approaches the Schwarzschild limit (figure adapted from (Chirenti et al., 2020)).

results shown in Fig. 1, but we also include the fit, which we model as an exponentially decaying function in the form

$$k_2 = a \left[1 - e^{-b(C-0.5)} \right] + d, \quad (18)$$

where the fitted coefficients are shown in the label of Fig. 2. We found that the R-squared value for this fitting model is $R^2 = 0.999429$, which shows that the exponentially decaying model is a good one. Note that in reference (Chirenti et al., 2020), this fitting was overlooked, so here we are complementing those results.

Finally, in Fig. 3 we show in log-scale the tidal Love number as a function of the compactness, and we also include the Post-Minkowskian (PM) expansion for constant density stars introduced by Chan et al. (2015). Note that for configurations with low compactness (Newtonian limit), the PM approximation is in good agreement with our results. However, when the compactness approaches the Buchdahl limit, the differences between both results increase. This difference is expected, considering that near the Buchdahl limit, we are entering into the strong gravity zone. Thus, the PM expansion is not a good approximation; in the strong gravity regime, we require the full general relativistic computation. In reference

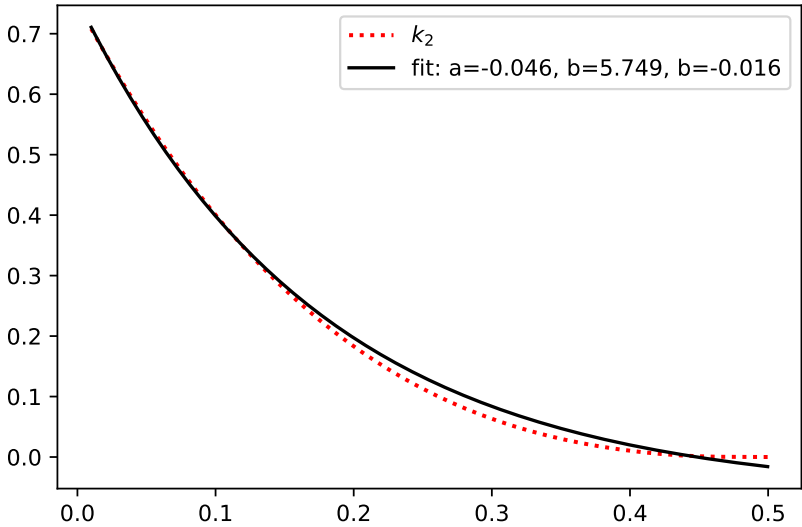


Figure 2. Same results as in Fig. 1, but we also present the fit in a black solid curve. We consider an exponentially decaying model in the form: $a \left[1 - e^{-b(C-0.5)} \right] + d$. The corresponding coefficients are shown in the label.

(Chirenti et al., 2020), figure 4 shows a similar log-scale for k_2 , together with the PM approximation. After the publication of that paper, the author found that the plot was marred due to a typo in the PM expansion. However, this error does not affect the results of k_2 .

4 DISCUSSION

The Schwarzschild interior solution with constant density, or Schwarzschild star, remains an interesting and simple example of an exact solution to Einstein’s equations for a perfect fluid. One of the most remarkable features of this model only discovered recently, is that it allows for compactness to cross beyond the Buchdahl limit. Moreover, these configurations can approach arbitrarily close to the Schwarzschild radius, where they become essentially the gravastar.

It has been found that the Love number k_2 of the ultracompact Schwarzschild stars is a smooth and continuous function of the compactness, and it approaches zero as the compactness approaches the BH limit 1/2. Thus, one can conclude that the result $k_2 \rightarrow 0$ is not an exclusive property of BHs but rather a consequence of the compactness approaching the Schwarzschild limit.

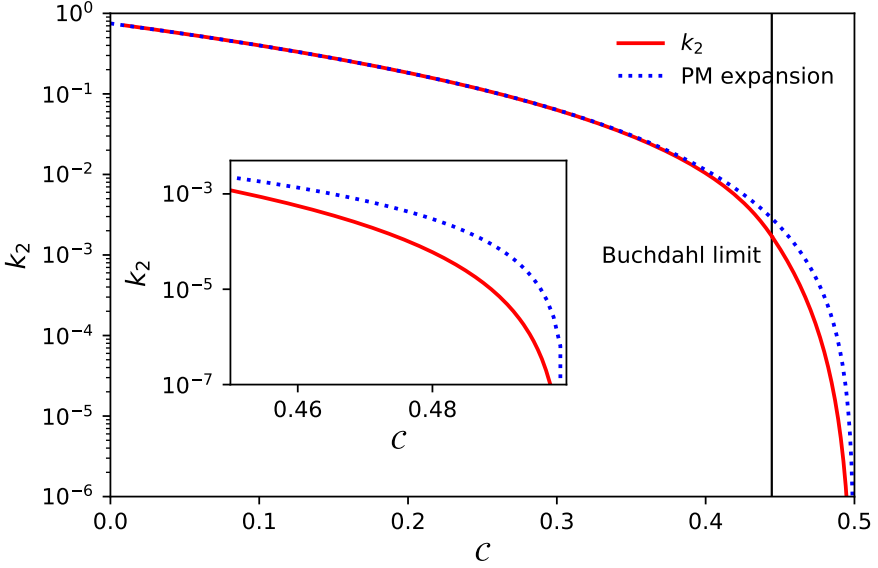


Figure 3. Same as Fig. 1, but in log-scale to emphasise the approach to the Schwarzschild limit. For comparison, we also show the Post-Minkowskian approximation for constant density stars presented in (Chan et al., 2015). In the Newtonian limit, both approaches are in good agreement. In the strong gravity zone, near the Buchdahl limit, the deviations start to grow as expected.

We found that the Love number of ultracompact Schwarzschild stars does not follow the logarithmic behaviour conjectured by (Cardoso et al., 2017) for generic ECOS. Instead, we found that k_2 decreases rapidly with compactness, following an exponentially decaying behaviour. Therefore, we believe that the conclusions drawn by (Cardoso et al., 2017), regarding the behaviour of k_2 for ECOS, are too restrictive considering that they studied only 4 types of ECOS, namely, thin-shell gravastar, boson stars, wormholes and some kind of anisotropic stars.

ACKNOWLEDGEMENTS

The author acknowledges the support of the Institute of Physics in Opava and its Research Centre for Theoretical Physics and Astrophysics.

REFERENCES

Beltracchi, P. and Gondolo, P. (2019), An exact time-dependent interior Schwarzschild solution, *Phys. Rev. D*, **99**(8), p. 084021, arXiv: 1902.07827.

- Beltracchi, P., Gondolo, P. and Mottola, E. (2021), Slowly rotating gravastars, arXiv: 2107.00762.
- Binnington, T. and Poisson, E. (2009), Relativistic theory of tidal Love numbers, *Phys. Rev. D*, **80**, p. 084018, arXiv: 0906.1366.
- Buchdahl, H. A. (1959), General Relativistic Fluid Spheres, *Phys. Rev.*, **116**, pp. 1027–1034.
- Cardoso, V., Franzin, E., Maselli, A., Pani, P. and Raposo, G. (2017), Testing strong-field gravity with tidal Love numbers, *Phys. Rev. D*, **95**(8), p. 084014, [Addendum: Phys.Rev.D 95, 089901 (2017)], arXiv: 1701.01116.
- Cardoso, V. and Pani, P. (2019), Testing the nature of dark compact objects: a status report, *Living Rev. Rel.*, **22**(1), p. 4, arXiv: 1904.05363.
- Celotti, A., Miller, J. C. and Sciamma, D. W. (1999), Astrophysical evidence for the existence of black holes: Topical review, *Class. Quant. Grav.*, **16**, p. A3, arXiv: astro-ph/9912186.
- Chan, T. K., Chan, A. P. O. and Leung, P. T. (2015), I-Love relations for incompressible stars and realistic stars, *Phys. Rev.*, **D91**(4), p. 044017, arXiv: 1411.7141.
- Chandrasekhar, S. (1985), *The mathematical theory of black holes*, Oxford University Press.
- Chandrasekhar, S. and Miller, J. C. (1974), On slowly rotating homogeneous masses in general relativity, *Mon. Not. Roy. Astron. Soc.*, **167**, p. 63.
- Charalambous, P., Dubovsky, S. and Ivanov, M. M. (2021), On the Vanishing of Love Numbers for Kerr Black Holes, *JHEP*, **05**, p. 038, arXiv: 2102.08917.
- Chia, H. S. (2021), Tidal deformation and dissipation of rotating black holes, *Phys. Rev. D*, **104**(2), p. 024013, arXiv: 2010.07300.
- Chirenti, C., Posada, C. and Guedes, V. (2020), Where is Love? Tidal deformability in the black hole compactness limit, *Class. Quant. Grav.*, **37**(19), p. 195017, arXiv: 2005.10794.
- Damour, T. and Nagar, A. (2009), Relativistic tidal properties of neutron stars, *Phys. Rev.*, **D80**, p. 084035, arXiv: 0906.0096.
- Glendenning, N. K. (2000), *Compact stars: Nuclear physics, particle physics, and general relativity*, Springer, 2nd. edition.
- Harrison, B. K., Thorne, K. S., Wakano, M. and Wheeler, J. A. (1965), *Gravitation theory and gravitational collapse*, The University of Chicago Press.
- Hartle, J. B. (1967), Slowly rotating relativistic stars. I. Equations of structure, *Astrophys. J.*, **150**, pp. 1005–1029.
- Hartle, J. B. and Thorne, K. S. (1968), Slowly Rotating Relativistic Stars. II. Models for Neutron Stars and Supermassive Stars, *Astrophys. J.*, **153**, p. 807.
- Hawking, S. W. and Ellis, G. F. R. (1973), *The Large Scale Structure of Space-Time*, Cambridge Monographs on Mathematical Physics, Cambridge University Press.
- Hinderer, T. (2008), Tidal Love numbers of neutron stars, *Astrophys. J.*, **677**, pp. 1216–1220, arXiv: 0711.2420.
- Hui, L., Joyce, A., Penco, R., Santoni, L. and Solomon, A. R. (2021), Static response and Love numbers of Schwarzschild black holes, *JCAP*, **04**, p. 052, arXiv: 2010.00593.
- Konoplya, R. A., Posada, C., Stuchlík, Z. and Zhidenko, A. (2019), Stable Schwarzschild stars as black-hole mimickers, *Phys. Rev. D*, **100**(4), p. 044027, arXiv: 1905.08097.
- Mazur, P. O. and Mottola, E. (2001), Gravitational condensate stars: An alternative to black holes, arXiv: gr-qc/0109035.
- Mazur, P. O. and Mottola, E. (2004), Gravitational vacuum condensate stars, *Proc. Nat. Acad. Sci.*, **101**, pp. 9545–9550, arXiv: gr-qc/0407075.
- Mazur, P. O. and Mottola, E. (2015), Surface tension and negative pressure interior of a non-singular ‘black hole’, *Class. Quant. Grav.*, **32**(21), p. 215024, arXiv: 1501.03806.

- Ovalle, J., Posada, C. and Stuchlík, Z. (2019), Anisotropic ultracompact Schwarzschild star by gravitational decoupling, *Class. Quant. Grav.*, **36**(20), p. 205010, arXiv: 1905.12452.
- Poisson, E. (2021), Tidally induced multipole moments of a nonrotating black hole vanish to all post-Newtonian orders, *Phys. Rev. D*, **104**(10), p. 104062, arXiv: 2108.07328.
- Posada, C. (2017), Slowly rotating supercompact Schwarzschild stars, *Mon. Not. Roy. Astron. Soc.*, **468**(2), pp. 2128–2139, arXiv: 1612.05290.
- Posada, C. and Chirenti, C. (2019), On the radial stability of ultra compact Schwarzschild stars beyond the Buchdahl limit, *Class. Quant. Grav.*, **36**, p. 065004, arXiv: 1811.09589.
- Postnikov, S., Prakash, M. and Lattimer, J. M. (2010), Tidal Love Numbers of Neutron and Self-Bound Quark Stars, *Phys. Rev. D*, **82**, p. 024016, arXiv: 1004.5098.
- Wald, R. M. (2001), The thermodynamics of black holes, *Living Rev. Rel.*, **4**, p. 6, arXiv: gr-qc/9912119.