Proceedings of *RAGtime 23–25*
Z. Stuchlík, G. Török, V. Karas and D. Lančová, editors, Silesian University in Opava, 2023, pp. 13–19

Astrophysical black holes embedded in organized magnetic fields

Case of a nonvanishing electric charge

Vladimír Karas

Astronomical Institute, Czech Academy of Sciences, Boční II 1401,
CZ-14100 Prague, Czech Republic
vladimir.karas@asu.cas.cz

ABSTRACT
Large-scale magnetic fields pervade the cosmic environment where the astrophysical black holes are often embedded and influenced by mutual interaction. In this contribution, we outline the appropriate mathematical framework to describe magnetized black holes within General Relativity, and we show several examples of how these can be employed in the astrophysical context. In particular, we examine the magnetized black hole metric in terms of an exact solution of electro-vacuum Einstein-Maxwell equations under the influence of a non-vanishing electric charge. New effects emerge: the expulsion of the magnetic flux out of the black-hole horizon depends on the intensity of the imposed magnetic field.

Keywords: Black holes – electromagnetic fields – general relativity

1 INTRODUCTION

Astrophysical black holes are cosmic objects that can be mathematically described by a set of Einstein-Maxwell equations (e.g., Romero and Vila, 2014). Various formulations of the Uniqueness Theorems express in a rigorous way the conditions under which the black hole solutions exist, and they constrain the parameter space that is necessary to specify different cases (Wald, 1984). It turns out that classical black holes are described by a small number of such parameters, in particular, the mass, electric (or magnetic) charge, and angular momentum (spin). Black holes do not support their own magnetic field except the gravito-magnetically induced components in the rotating, charged Kerr-Newman metric.

However, astrophysical black holes are embedded in a magnetic field of external origin, which then interacts with the internal properties of the black hole (Ruffini and Wilson, 1975). In the case of very strong magnetic intensity, the magnetic field even contributes to the spacetime metric. In the present contribution, we examine the interesting properties of such an electrically charged, magnetized, rotating black hole. To this end, we employ the solution originally derived in the 1970s by means of Ernst magnetization techniques

978-80-7510-577-6

(Ernst and Wild, 1976) and demonstrate its interesting features in terms of magnetic flux threading different regions of the black hole horizon or an entire hemisphere (see Bičák and Hejda (2015), and further references cited therein).

We limit our discussion to axially symmetric and stationary solutions. These are vacuum, asymptotically non-flat solutions, where the influence of plasma is ignored, but the effects of strong gravity are taken into account.

2 MAGNETIZED BLACK HOLES WITH SPIN AND CHARGE

We can write the system of mutually coupled Einstein-Maxwell partial differential equations (Chandrasekhar, 1983),

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1}$$

where the source term $T_{\mu\nu}$ is of purely electromagnetic origin,

$$T^{\alpha\beta} \equiv T^{\alpha\beta}_{\rm EMG} = \frac{1}{4\pi}\left(F^{\alpha\mu}F^{\beta}_{\mu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}g^{\alpha\beta}\right), \tag{2}$$

$$T^{\mu\nu}{}_{;\nu} = -F^{\mu\alpha} j_{\alpha}, \qquad F^{\mu\nu}{}_{;\nu} = 4\pi j^{\mu}, \qquad {}^{\star}F^{\mu\nu}{}_{;\nu} = 4\pi \mathcal{M}^{\mu}, \tag{3}$$

and ${}^{\star}F_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$. We will consider the spacetime solutions for the metric that satisfies the electro-vacuum case with a regular event horizon under the constraints of axial symmetry and stationarity,

$$\mathrm{d}s^2 = f^{-1}\left[e^{2\gamma}\left(\mathrm{d}z^2 + \mathrm{d}\rho^2\right) + \rho^2\,\mathrm{d}\phi^2\right] - f\,(\mathrm{d}t - \omega\,\mathrm{d}\phi)^2\,, \tag{4}$$

with f, ω, and γ being functions of z and ρ only. Hereafter, instead of the canonical cylindrical form (4), we will also employ the spheroidal coordinates r and θ when convenient. In the weak electromagnetic field approximation, the electromagnetic (test) field is supposed to reside in the background of a rotating black hole, e.g., Kerr metric or a weakly charged Kerr metric (e.g., Wald, 1984; Gal'tsov, 1986). As an example, in such an asymptotically flat spacetime, the axial Killing vector $\partial_\phi(\equiv\tilde{\xi})$ generates a uniform magnetic field, whereas the field vanishes asymptotically for the time-like Killing vector $\partial_t(\equiv\xi)$. These two solutions are known as the Wald's field, i.e.

$$\boldsymbol{F} = \tfrac{1}{2}B_0\left(\mathbf{d}\tilde{\xi} + \tfrac{2J}{M}\,\mathbf{d}\xi\right) \tag{5}$$

in Wald (1974) notation. Magnetic flux surfaces are defined,

$$4\pi\Phi_{\rm m} = \int_S \boldsymbol{F} \;=\; \text{const.} \tag{6}$$

Magnetic and electric (Lorentz) forces are then given in terms of the particle's mass m, its four-velocity $\boldsymbol{u}$, and magnetic and electric charges, $q_{\rm m}$ and $q_{\rm e}$, respectively:

$$m\dot{\boldsymbol{u}} = q_{\rm m}{}^{\star}\boldsymbol{F}.\boldsymbol{u}, \qquad m\dot{\boldsymbol{u}} = q_{\rm e}\boldsymbol{F}.\boldsymbol{u}. \tag{7}$$

Magnetic field lines are determined by

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{B_r}{B_\theta}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{B_r}{B_\phi}, \tag{8}$$

in a perfect analogy with classical electromagnetism. We will employ the above-given quantities in our discussion further below.

Magnetic (electric) lines of force are defined by the direction of Lorentz force that acts on electric (magnetic) charges,

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \propto {}^\star F^\mu_\nu\, u^\nu, \qquad \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \propto F^\mu_\nu\, u^\nu. \tag{9}$$

Therefore, in an axially symmetric system, the equation for magnetic lines takes a lucid form,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{F_{\theta\phi}}{F_{r\phi}}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{F_{\theta\phi}}{F_{r\theta}}, \tag{10}$$

that is in correspondence with eq. (8).

Let us now turn our attention to the case of a strong magnetic field, where we cannot ignore its influence on the spacetime metric. The latter is not necessarily flat in the asymptotical spatial region far from the black hole (Ernst and Wild, 1976; Karas and Vokrouhlický, 1991).

As an example, we can consider magnetized Kerr-Newman black hole metric expressed in the form (García Díaz, 1985)

$$\mathrm{d}s^2 = |\Lambda|^2\, \Sigma \left(\Delta^{-1}\, \mathrm{d}r^2 + \mathrm{d}\theta^2 - \Delta A^{-1}\, \mathrm{d}t^2\right) + |\Lambda|^{-2}\, \Sigma^{-1} A \sin^2\theta\, (\mathrm{d}\phi - \omega\, \mathrm{d}t)^2\,, \tag{11}$$

where $\Sigma = r^2{+}a^2\cos^2\theta$, $\Delta = r^2{-}2Mr{+}a^2{+}e^2$, $A = \left(r^2 + a^2\right)^2 {-}\Delta a^2 \sin^2\theta$ are functions from the Kerr-Newman metric. The outer horizon is located at radius $r{\equiv}r_+ = 1{+}\left(1 - a^2 - e^2\right)^{1/2}$, like in an unmagnetized case, and the horizon existence is restricted to the range of parameters $a^2{+}e^2 \leq 1$. Let us emphasise that, in the magnetized case, the traditional Kerr-Newman parameters a and e are *not identical* with the black hole total spin and electric charge, as we will see further below. Moreover, because of the asymptotically non-flat nature of the spacetime, the Komar-type angular momentum and electric charge (as well as the black hole mass) have to be defined by integration over the horizon sphere rather than at radial infinity.

The magnetization function $\Lambda = 1 + \beta\Phi - \frac{1}{4}\beta^2\mathcal{E}$, in terms of the Ernst potentials $\Phi\,(r,\theta)$ and $\mathcal{E}\,(r,\theta)$, reads

$$\Sigma\Phi = ear\sin^2\theta - \mathrm{i}e\left(r^2 + a^2\right)\cos\theta, \tag{12}$$

$$\begin{aligned}\Sigma\mathcal{E} = &- A\sin^2\theta - e^2\left(a^2 + r^2\cos^2\theta\right)\\ &+ 2\mathrm{i}a\left[\Sigma\left(3 - \cos^2\theta\right) + a^2\sin^4\theta - re^2\sin^2\theta\right]\cos\theta.\end{aligned} \tag{13}$$

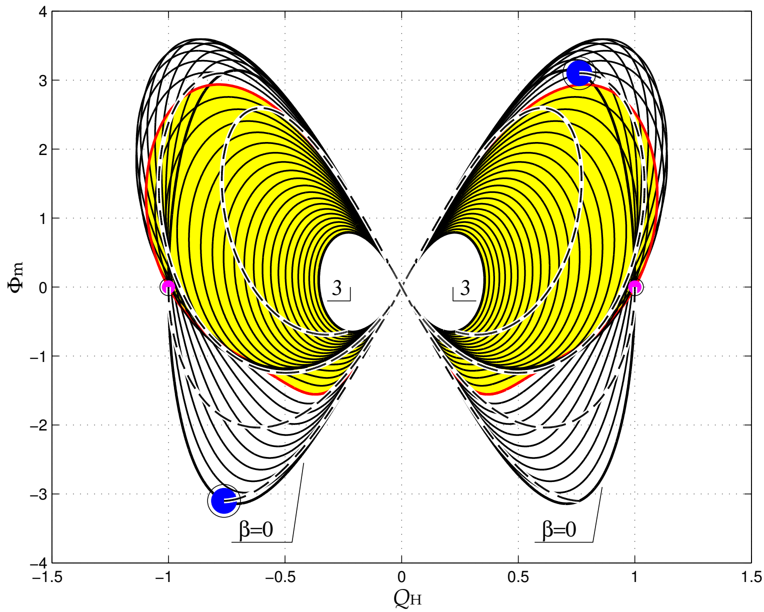

Figure 1. The "butterfly diagram" shows the magnetic flux $\Phi_{\rm m}$ of magnetized Kerr-Newman black hole with $a^2+e^2=1$ as a function of the total electric charge $Q_{\rm H}$. Solid curves correspond to a constant value of the dimensionless magnetization parameter $\beta=BM$ ($\beta=0$ is the case of an unmagnetized Kerr-Newman black hole). The area of the plot with ultra-strong magnetization is bounded by $\beta=1$ (red curve) and emphasized by yellow colour in the plot. The lines of a constant ratio of a/e and varying β are also plotted (dashed; the cases of $a/e=\pm0.85$ and 0 are shown); some distinctive combinations of the parameters a, e are emphasized by colour points.

The corresponding components of the electromagnetic field can be written conveniently with respect to orthonormal LNRF components,

$$H_{(r)}+{\rm i}E_{(r)}=A^{-1/2}\sin^{-1}\theta\,\Phi'_{,\theta}, \tag{14}$$
$$H_{(\theta)}+{\rm i}E_{(\theta)}=-(\Delta/A)^{1/2}\sin^{-1}\theta\,\Phi'_{,r}, \tag{15}$$

where $\Phi'(r,\theta)=\Lambda^{-1}\left(\Phi-\frac{1}{2}\beta\mathcal{E}\right)$. The total electric charge $Q_{\rm H}$ is

$$Q_{\rm H}=-\left|\Lambda_0\right|^2\;\Im{\rm m}\,\Phi'\left(r_+,0\right), \tag{16}$$

and the magnetic flux $\Phi_{\rm m}(\theta)$ across a cap placed in an axisymmetric position on the horizon is

$$\Phi_{\rm m}=2\pi\left|\Lambda_0\right|^2\;\Re{\rm e}\,\Phi'\left(r_+,\bar\theta\right)\Big|_{\bar\theta=0}^{\theta}, \tag{17}$$

where $\Lambda_0=\Lambda(\theta=0)$. The adopted notation indicates subtraction of the Ernst potential values $\Phi'\left(r_+,\bar\theta\right)$ taken at $\bar\theta\to\theta$ and $\bar\theta\to0$.

At this point, it is interesting to mention that the span of the azimuthal coordinate in the magnetized solution must be rescaled by the multiplication factor Λ_0 in order to avoid a conical singularity on the symmetry axis (Hiscock, 1981):

$$\Lambda_0 = \left[1 + \tfrac{3}{2}\beta^2 e^2 + 2\beta^3 ae + \beta^4 \left(\tfrac{1}{16}e^4 + a^2\right)\right]^{1/2}. \tag{18}$$

This rescaling procedure effectively leads to the increase of the horizon surface area, and thereby also magnetic flux across the horizon (Karas, 1988). In Figure 1, the magnetic flux across the entire black hole hemisphere in Kerr-Newman strongly magnetized black hole solution, $\Phi_{\rm m}(\theta = \pi/2)$, is shown as a function of electric charge on the horizon, $Q_{\rm H}$ (additional details can be found in Karas and Budinová (2000); Karas et al. (2019)).

Cases of intersection of the $\beta = $ const curves with $F = 0$ and non-zero charge, $Q_{\rm H} \neq 0$, correspond to the vanishing angular momentum of the black hole, $J = 0$. This property is rather different from the behaviour of weakly magnetized black holes with only a test magnetic field imposed on them. On the other hand, this exact solution does not allow us to study the effects of the misalignment of the magnetic field with respect to the rotation axis, which is so far possible only in the test-field approximation or by numerical techniques. Let us also note that it may be interesting to consider the magnetic flux also in other spacetime metrics for comparison and better understanding of the underlying processes (see, e.g., Gutsunaev and Manko (1988); Kovář et al. (2012); Khan and Chen (2023); Vrba et al. (2023), and further references cited therein).

3 CONCLUSIONS

We discussed the magnetic flux across the event horizon of a magnetized rotating black hole and the associated electric charge within the framework of the *exact solution* of mutually coupled Einstein-Maxwell equations. To this end, we adopted Ernst and Wild (1976) spacetime, which represents an axially symmetric, stationary solution that corresponds, in a straightforward manner, to the Wald (1974) solution of an asymptotically uniform magnetic field imposed on the background of Kerr metric. On the other hand, in the limit of small black hole mass, zero angular momentum and strong magnetic intensity, the adopted solution goes over to the cosmological solution of Melvin universe (Melvin, 1964). The latter represents an asymptotically non-flat 'geon' that is maintained in the static configuration by the gravitational effect of the magnetic field acting upon itself. In other words, this means that our discussion is appropriate for the limit of ultra-strong magnetic fields that might be possibly relevant in the conditions of early Universe (Beskin et al., 2015). On the other hand, we think that this extreme situation does not bring any qualitatively new phenomena into the discussion of the magnetic Penrose process, which has been recently discussed in the context of weakly magnetized black holes (Karas and Stuchlík, 2023).

We elaborated a detailed graphical representation of magnetic flux which exhibits an intricate structure as a function of the space-time parameters. This has allowed us to position specific configurations, such as those with vanishing angular momentum or vanishing electric charge. The corresponding combinations of the model parameters are different from the case of weak magnetic field limit because of strong-gravity effects. Let us note that ultra-strong magnetic fields are expected to affect processes on molecular and atomic scales in the conditions when the magnetic energy density becomes comparable to the binding energy density (Lai, 2015).

ACKNOWLEDGEMENTS

The author thanks the referee for careful reading of the original version of the manuscript and for helpful suggestions towards improvements at several points. The author acknowledges continued support from the Czech Science Foundation EXPRO grant titled "Accreting black holes in the new era of X-ray polarimetry missions", No. 21-06825X.

REFERENCES

Beskin, V. S., Balogh, A., Falanga, M. and Treumann, R. A. (2015), Magnetic fields at largest universal strengths: Overview, *Space Science Reviews*, **191**(1-4), pp. 1–12.

Bičák, J. and Hejda, F. (2015), Near-horizon description of extremal magnetized stationary black holes and meissner effect, *Physical Review D*, **92**(10), p. 104006.

Chandrasekhar, S. (1983), *The Mathematical Theory of Black Holes*, Oxford University Press.

Ernst, F. J. and Wild, W. J. (1976), Kerr black holes in a magnetic universe, *Journal of Mathematical Physics*, **17**(2), pp. 182–184.

Gal'tsov, D. V. (1986), *Particles and Fields around Black Holes*, Moscow University Press.

García Díaz, A. (1985), Magnetic generalization of the kerr–newman metric, *Journal of Mathematical Physics*, **26**(1), pp. 155–156.

Gutsunaev, T. I. and Manko, V. S. (1988), On a family of solutions of the einstein-maxwell equations, *General Relativity and Gravitation*, **20**(4), pp. 327–335.

Hiscock, W. A. (1981), On black holes in magnetic universes, *Journal of Mathematical Physics*, **22**(8), pp. 1828–1833.

Karas, V. (1988), Magnetic fluxes across black holes. exact models, *Bulletin of the Astronomical Institute of Czechoslovakia*, **39**.

Karas, V. and Budinová, Z. (2000), Magnetic fluxes across black holes in a strong magnetic field regime, *Physica Scripta*, **61**(2), pp. 253–256.

Karas, V. and Stuchlík, Z. (2023), Magnetized black holes: Interplay between charge and rotation, *Universe*, **9**(6), p. 267.

Karas, V., Svoboda, J. and Zajacek, M. (2019), Selected chapters on active galactic nuclei as relativistic systems, *arXiv preprint arXiv:1901.06507*.

Karas, V. and Vokrouhlický, D. (1991), On interpretation of the magnetized kerr–newman black hole, *Journal of Mathematical Physics*, **32**(3), pp. 714–716.

Khan, S. U. and Chen, Z.-M. (2023), Charged particle dynamics in black hole split monopole magnetosphere, *The European Physical Journal C*, **83**(8).

Kovář, J., Kopáček, O., Karas, V. and Kojima, Y. (2012), Regular and chaotic orbits near a massive magnetic dipole, *Classical and Quantum Gravity*, **30**(2), p. 025010.

Lai, D. (2015), Physics in very strong magnetic fields, *Space Science Reviews*, **191**(1-4), pp. 13–25.

Melvin, M. (1964), Pure magnetic and electric geons, *Physics Letters*, **8**(1), pp. 65–68.

Romero, G. E. and Vila, G. S. (2014), *Introduction to Black Hole Astrophysics*, Springer Berlin Heidelberg.

Ruffini, R. and Wilson, J. R. (1975), Relativistic magnetohydrodynamical effects of plasma accreting into a black hole, *Physical Review D*, **12**(10), pp. 2959–2962.

Vrba, J., Rayimbaev, J., Stuchlik, Z. and Ahmedov, B. (2023), Charged particles motion and quasiperiodic oscillation in simpson–visser spacetime in the presence of external magnetic fields, *The European Physical Journal C*, **83**(9).

Wald, R. M. (1974), Black hole in a uniform magnetic field, *Physical Review D*, **10**(6), pp. 1680–1685.

Wald, R. M. (1984), *General Relativity*, University of Chicago Press.