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Charged particles on resonant orbits around Schwarzschild black hole

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ABSTRACT
We explore the dynamics of test particles on perturbed circular orbits in the equatorial plane of a Schwarzschild black hole in search of resonant effects. The nonlinear bond between radial and vertical oscillatory modes is given by Lorentz electromagnetic force acting on charged particles in the uniform magnetic field. When the perturbation of the circular orbit is large enough, strong, persistent 2:1 resonance between radial and vertical modes develops.

Keywords: Black hole – resonances – particle dynamics – magnetic field

1 INTRODUCTION

The microquasars are binary systems composed of a black hole (BH) and a companion (donor) star, and Quasi-Periodic Oscillations (QPOs) are periodic changes in X-ray photons flux in these systems (Remillard and McClintock, 2006). QPOs cover a wide range of frequencies, from low-frequency QPOs (~ 30 Hz) to high-frequency QPOs (~ 500 Hz). The QPOs are still unresolved phenomena, but the connection to particle orbital frequency at the innermost stable circular orbit (ISCO) (~200 Hz for 10M$_\odot$ BH) is frequently assumed (Török et al., 2005). Most high-frequency QPOs in BHs are detected with twin peaks with a frequency ratio ~3:2; obviously, some resonance phenomena in the BH accretion disk are present. In this proceeding, we will simulate accretion disk resonances using simple charged test particle dynamics as a model for plasma around BH.

2 CHARGED PARTICLE DYNAMICS

We consider a BH of a mass M described by the Schwarzschild metric

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f^{-1}(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \qquad f(r) = 1 - \frac{2M}{r}, \tag{1}$$

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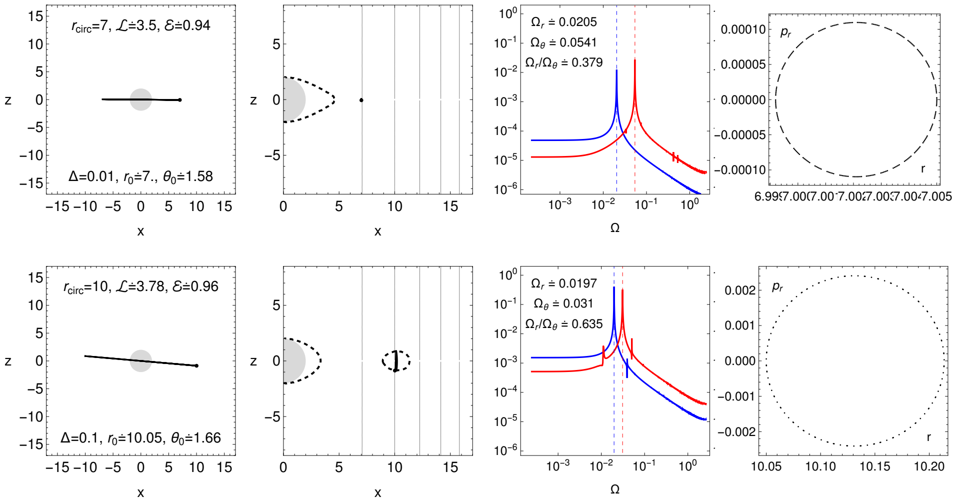

Figure 1. Charged particle oscillations around circular orbit with very weak magnetic field influence $\mathcal{B} = 10^{-5}$. We see particle trajectories in x-z plane (first column) with BH as a grey disk, projection in x-z plane (second column) with dashed energetic boundary, Fourier spectra (third column) for radial r and vertical θ coordinates with frequency ratios, dashed vertical lines pointing maxima, Poincaré surface of section (fourth column) for equatorial plane crossing. The first row of figures is for small perturbation $\triangle = 0.01$ from circular orbit, where only main frequencies are present, and the second row is for larger perturbation $\triangle = 0.1$, where we can see higher harmonic in the spectra.

where $f(r)$ is the lapse function. Let the BH be immersed into an external uniform magnetic field, given by electromagnetic four-potential (Wald, 1974)

$$A_\phi = \frac{B}{2} r^2 \sin^2\theta. \tag{2}$$

Hereafter, we put $M = 1$, i.e., we use dimensionless radial coordinate r (and time coordinate t).

The equations of motion for a charged particle with mass m and electric charge q can be obtained using the Hamiltonian formalism

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\zeta} = \frac{\partial H}{\partial \pi_\mu}, \quad \frac{\mathrm{d}\pi_\mu}{\mathrm{d}\zeta} = -\frac{\partial H}{\partial x^\mu}, \qquad H = \frac{1}{2} g^{\alpha\beta}(\pi_\alpha - qA_\alpha)(\pi_\beta - qA_\beta) + \frac{m^2}{2} = 0, \tag{3}$$

where the kinematical four-momentum $p^\mu = mu^\mu = \mathrm{d}x^\mu/\mathrm{d}\zeta$ is related to the generalized (canonical) four-momentum π^μ by the relation $\pi^\mu = p^\mu + qA^\mu$. The affine parameter ζ of the particle is related to its proper time τ by the relation $\zeta = \tau/m$.

Due to the symmetries of the Schwarzschild spacetime (1) and the magnetic field (2), one can easily find the conserved quantities that are particle energy and axial angular momentum and magnetic field parameters $\mathcal{B}$

$$\mathcal{E} = \frac{E}{m} = -\frac{\pi_t}{m} = -g_{tt}u^t, \quad \mathcal{L} = \frac{L}{m} = \frac{\pi_\phi}{m} = g_{\phi\phi}u^\phi + \frac{q}{m}A_\phi, \qquad \mathcal{B} = \frac{qB}{2m}, \tag{4}$$

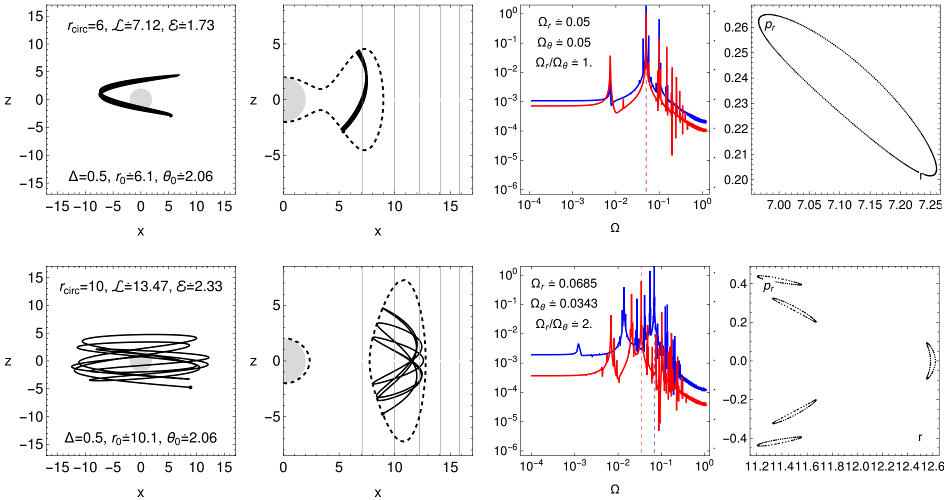

Figure 2. Charged particle oscillations around circular orbit in stronger magnetic field influence $\mathcal{B} = 0.1$, similar to Fig. 1. In the first row of figures, the motion has complicated Fourier spectra, but the main frequency peaks are in resonant ratio $\Omega_r/\Omega_\theta \sim 1$. The trajectory in the second row is close to chaotic motion, but the main peak frequencies are still in resonant ratio $\Omega_r/\Omega_\theta \sim 2$, which appear frequently in our numerical experiment.

which reflects a relative relationship between Lorentz and gravitational forces. Using such symmetries, one can rewrite the Hamiltonian (3) in the form

$$H = \frac{1}{2}g^{rr}p_r^2 + \frac{1}{2}g^{\theta\theta}p_\theta^2 + \frac{1}{2}g^{tt}E^2 + \frac{1}{2}g^{\phi\phi}\left(L - qA_\phi\right)^2 + \frac{1}{2}m^2. \tag{5}$$

Energetic boundary for particle motion can be expressed from the equation (5)

$$\mathcal{E}^2 = V_{\rm eff}\,(r,\theta) \quad (\text{for } p_r = p_\theta = 0)\,. \tag{6}$$

We introduced effective potential for charged particles $V_{\rm eff}(r,\theta)$ by the relation

$$V_{\rm eff}\,(r,\theta) \equiv -g_{tt}\left[g^{\phi\phi}\left(\mathcal{L} - \frac{q}{m}A_\phi\right)^2 + 1\right]. \tag{7}$$

The effective potential $V_{\rm eff}\,(r,\theta)$ combines the influence of gravity potential (g_{tt} term) with the influence of central force potential given by the specific angular momentum $\mathcal{L}$ and electromagnetic potential energy given by qA_ϕ.

A detailed description of charged particle dynamics around BH can be found for the uniform magnetic field in Galtsov and Petukhov (1978); Karas and Vokrouhlicky (1990); Frolov and Shoom (2010); Kološ et al. (2015); Kopáček and Karas (2018) or for more realistic parabolic BH magnetosphere in Kološ et al. (2023).

If a charged test particle is slightly displaced from the equilibrium position located in a minimum of the effective potential $V_{\rm eff}\,(r,\theta)$ at r_0 and $\theta_0 = \pi/2$, corresponding to a stable

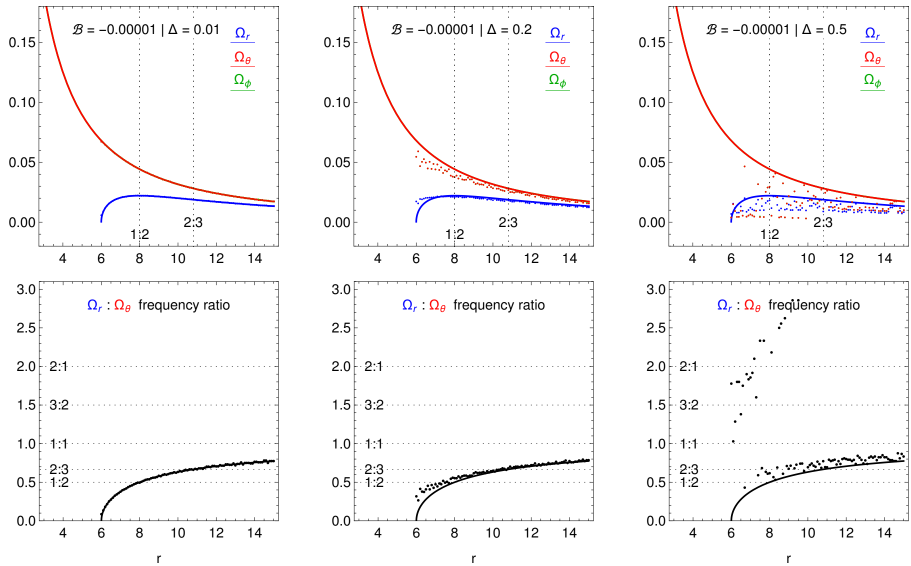

Figure 3. Test particle orbital frequencies (first row), and the ratio of radial and vertical frequencies (second row) with weak electromagnetic influence ($\mathcal{B} = -10^{-5}$). For small circular orbit perturbation $\triangle = 0.01$, the numerically calculated frequencies (individual dots) from perturbed orbit exactly follow the analytic frequencies (solid curves) given by effective potential minima. For large perturbation $\triangle = 0.5$, the numerical frequencies are detached from the analytical one. Different resonant radii (vertical dotted lines in the first row and horizontal dotted lines in the second row) are plotted, but no clustering around them can be reported.

circular orbit, the particle will start to oscillate around the minimum realising thus epicyclic motion governed by linear harmonic oscillations with the radial Ω_r, vertical Ω_θ, and axial Ω_ϕ frequencies (Wald, 1984)

$$\Omega_r^2 = \frac{1}{2}\frac{f^2(r)}{\mathcal{E}^2(r)}\frac{\partial^2 V_{\rm eff}}{\partial r^2}, \quad \Omega_\theta^2 = \frac{1}{2}\frac{f(r)}{r^2\mathcal{E}^2(r)}\frac{\partial^2 V_{\rm eff}}{\partial \theta^2}, \quad \Omega_\phi^2 = \frac{\mathcal{L}^2(r)}{g_{\phi\phi}^2}\frac{f^2(r)}{\mathcal{E}^2(r)}. \tag{8}$$

3 RESONANCES FOR PERTURBED CIRCULAR ORBIT

A thin Keplerian accretion disk model is given by a dense set of particles on a circular orbit where gravity is perfectly compensated by centrifugal force. In our case of charged test particles, the Lorentz force from an external uniform magnetic field will also be taken into account. To simulate resonances in the accretion disk around BH, we randomly perturb all particles on the circular orbits in radial δr and vertical direction $\delta\theta$. For all different circular

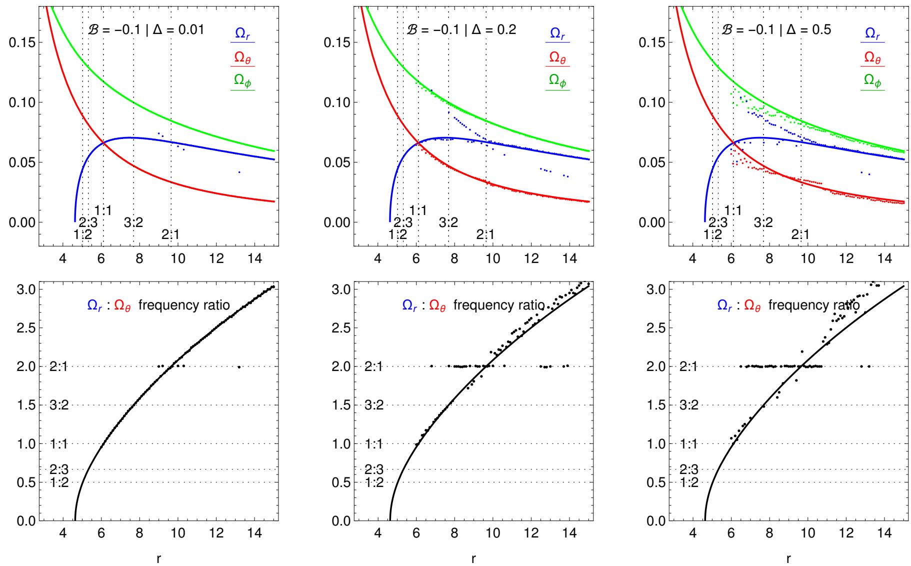

Figure 4. Charged particle orbital frequencies (first row) and the frequency ratio (second row) with stronger magnetic field influence $\mathcal{B} = -0.1$, similar to Fig. 3. When the circular orbit is only slightly perturbed $\triangle = 0.1$, the numerical and analytical frequencies coincide for almost all the points with only a few 2:1 resonant exceptions. The number of trajectories with a 2:1 resonant ratio grows with circular orbit perturbation, reaching more than 50% for $\triangle = 0.5$ value.

orbits from the Keplerian disk, the total perturbation $\triangle$ will remain the same

$$r = r_{\text{circ}} + \delta r, \quad \theta = \pi/2 + \delta\theta, \qquad \triangle = \sqrt{\delta r^2 + \delta\theta^2}. \tag{9}$$

As it has been demonstrated in Fig. 3, when the perturbation δ is small, the main peak frequency calculated from particle trajectory using Fourier transform coinciding with analytically calculated frequencies for small circular orbit perturbation (8). When the perturbation is large enough and when the nonlinear electromagnetic bound between r and θ oscillatory modes is present, the particle trajectory main frequency peaks are likely to be in resonant ratios 2:1, see Fig. 4.

While resonant ratios between oscillatory modes might appear coincidental for a single particle orbit, they become statistically significant when considering the dense set of circular orbits covering the entire Keplerian disk. When perturbed, the charged particles are likely to orbit around BH in 2:1 resonance between $\Omega_r : \Omega_\theta$ modes as demonstrated in Figs. 3 and 4.

The parametric resonance model (Abramowicz and Kluźniak, 2004) is given by formula

$$\frac{\Omega_r}{\Omega_\theta} = \frac{2}{n}, \quad n = 1, 2, 3, \ldots, \tag{10}$$

where the strongest resonances can be expected for small n. For the observed QPOs signal, a 3:2 resonance has been reported and not a 2:1 resonance. However, this discrepancy could be explained by the fact that we do not observe Ω_r and Ω_θ directly, but we observe their combinations

$$\Omega_r : \Omega_\theta = 2 : 1 \quad \rightarrow \quad \Omega_r + \Omega_\theta : \Omega_r = 3 : 2. \tag{11}$$

We currently have no direct explanation for the 2:1 resonance and why we do not observe stronger 1:1 or weaker 3:2 resonances, even though they are allowed in our model for charged particle dynamics. The effective potential for a charged particle has a special shape, it has Z_2 symmetry ($V_{\rm eff}(x, z)$ is the same above $z > 0$ and below $z < 0$ the equatorial plane), which could be one of the possible explanations for the observed strong 2:1 resonance. Another explanation could come from the Kolmogorov-Arnold-Moser (KAM) theorem. If we can express our system as having the regular part H_0 and a perturbative part $H_{\rm p}$, with perturbation parameter ϵ, then the Hamiltonian can be expressed as

$$H = H_0 + \epsilon H_{\rm p}. \tag{12}$$

According to the KAM theorem, only nonresonant tori will survive small ϵ perturbation, while the resonant tori will be destroyed. For two degrees of freedom system, as is our particle dynamics in an axially symmetric model, one will have

$$k_1\,\Omega_r + k_2\,\Omega_\theta = 0, \quad k_1 + k_2 < 4, \tag{13}$$

and hence around the resonant elliptic point (minima in effective potential), we can not construct Birkhoff normal form if $k_1 + k_2 > 4$ (Tabor, 1989). The condition (13) is correct only for 1:1, 1:2, and 2:1 resonances but not $3 : 2$ and could explain why 3:2 resonance between Ω_r and Ω_θ is not observed in our model.

4 CONCLUSIONS

In our numerical experiment with charged particle dynamics around magnetized Schwarzschild BH, we have demonstrated the existence of 2:1 resonance, which could be related to observed QPO within the parametric resonance model. It is still an open question, whether these 2:1 resonances are unique to our test particle model, and if 2:1 will appear in other systems. In the future, we would like to explore other nonlinear models, like string-loop, spinning particle dynamics, and test particle motion in modified BH spacetimes.

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