

# Can extended bodies follow geodesic trajectories?

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## ABSTRACT

We provide an extension of the analysis on whether an extended test body can follow a geodesic trajectory given by [Mukherjee et al. \(2022\)](#). In particular, we consider a test body in a pole-dipole-quadrupole approximation under the Ohashi-Kyrian-Semerák spin supplementary condition moving in the Schwarzschild and Kerr background. Using orbital setups under which a pole-dipole body can follow geodesic motion, we explore under which conditions this can also take place in the pole-dipole-quadrupole approximation when only the mass quadrupole is taken into account. For our analysis, we employ the assumption that the dipole contribution and the quadrupole contribution vanish independently.

**Keywords:** Mathisson–Papapetrou–Dixon equations – particle dynamics – black holes

## 1 INTRODUCTION

The framework of an extended test body moving on a curved background, which we employ, dates back to the pioneering works of [Mathisson \(1937\)](#) and [Papapetrou \(1951\)](#). A significant contribution to this framework was provided by [Dixon \(1970a,b, 1974\)](#), which resulted in calling the respective equation of motion the Mathisson-Papapetrou-Dixon (MPD) equations. These equations in the pole-dipole-quadrupole approximation, when only gravitational interactions are considered, read [Steinhoff and Puetzfeld \(2010, 2012\)](#)

$$\dot{P}^\mu = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}\mathcal{U}^\nu S^{\alpha\beta} - \frac{1}{6}J^{\alpha\beta\gamma\delta}\nabla^\mu R_{\alpha\beta\gamma\delta}, \quad (1)$$

$$\dot{S}^{\mu\nu} = 2P^{[\mu}\mathcal{U}^{\nu]} + \frac{4}{3}J^{\alpha\beta\gamma[\mu}R^{\nu]\gamma\alpha\beta}, \quad (2)$$

where  $R^\mu{}_{\nu\alpha\beta}$  is the Riemann tensor,  $S^{\alpha\beta}$  is the spin tensor,  $J^{\alpha\beta\gamma\delta}$  is the quadrupole tensor,  $P^\mu$  is the four-momentum and  $\mathcal{U}^\mu$  is the four-velocity, while the ‘dot’ defines a covariant

derivative with respect to the proper time. The MPD equations do not describe the evolution of the quadrupole moment, which has to be determined by the matter structure of the body (see, e.g., [Steinhoff and Puetzfeld, 2012](#)).

It is obvious from [Eq. \(1\)](#) that an extended body should, in general, deviate from a geodesic trajectory even if  $P^\mu \parallel \mathcal{U}^\mu$ . Note that  $P^\mu \parallel \mathcal{U}^\mu$  is not in general the case for the MPD equations. The question we tackled in [Mukherjee et al. \(2022\)](#) was whether these rules can have exceptions on a black hole background, i.e. can extended bodies follow geodesic trajectories? We have found some positive answers in the pole-dipole approximation but failed to do so in the pole-dipole-quadrupole in the spin-induced quadrupole case.

In the present work, we restrict our analysis to the contribution of the mass quadrupole moment  $Q^{\beta\gamma}$  to the quadrupole term. Then, the quadrupole tensor reads

$$J^{\alpha\beta\gamma\delta} = -3\mathcal{V}^{[\alpha} Q^{\beta][\gamma} \mathcal{V}^{\delta]}, \quad (3)$$

where  $\mathcal{V}$  is a future-oriented time-like vector, which is employed to fix the centre of the mass of the body by using the constraint

$$\mathcal{V}_\mu S^{\mu\nu} = 0. \quad (4)$$

This constraint is known in the literature as the spin supplementary condition (SSC). There are several SSCs (see, e.g., [Costa et al., 2018](#)), but in [Mukherjee et al. \(2022\)](#) as in this work we focus on the Ohashi-Kyrian-Semerák (OKS) one ([Ohashi, 2003](#); [Kyrian and Semerak, 2007](#)). For this SSC, a vector  $\mathcal{V}^\mu = w^\mu$  is chosen so that  $\dot{w}^\mu = 0$  and  $w^\mu w_\mu = -1$ .

This OKS choice leads to  $\dot{S}^{\mu\nu} w_\mu = 0$ ,  $\dot{S}^{\mu\nu} w_\mu = 0$  and all the contractions of  $w^\mu$  with the higher covariant derivatives of the spin tensor are also equal to zero. Hence, by contracting [Eq. \(2\)](#) with  $w_\mu$  leads to

$$P^\mu = \frac{1}{-w_\nu \mathcal{U}^\nu} \left[ (-P^\gamma w_\gamma) \mathcal{U}^\mu + K^{\mu\delta} w_\delta \right], \quad (5)$$

where we have set  $K^{\mu\nu} = 4/3 J^{\alpha\beta\gamma\mu} R^{\nu\lambda}_{\gamma\alpha\beta}$ . It is straightforward that if there is no quadrupole term, then the four-momentum and the four-velocity become parallel, which is a feature of geodesic orbits. [Mukherjee et al. \(2022\)](#) have shown that this feature of OKS SSC can be recovered even with the quadrupole term as long as

$$K^{\mu\gamma} w_\gamma = 0 \quad (6)$$

holds.

In the cases that  $P^\mu \parallel \mathcal{U}^\mu$ , it is said that the hidden momentum vanishes, since  $P^\mu$  can be split in a part parallel  $P^\mu_{\parallel}$  to  $\mathcal{U}^\mu$  and to the hidden momentum part  $P^\mu_{\text{hid}}$  ([Filipe Costa and Natário, 2014](#)) leading to

$$P^\mu = P^\mu_{\parallel} + P^\mu_{\text{hid}}. \quad (7)$$

The parallel part reads

$$P^\mu_{\parallel} = m \mathcal{U}^\mu, \quad (8)$$

where the mass  $m := -P_\mu \mathcal{U}^\mu$ . In the MPD formalism, there is also another mass defined as  $\mu^2 = -P^\mu P_\mu$ . Note that  $\mu$  and  $m$  in general do not coincide.

Our work concerns motion in a Kerr black hole background, for which the metric tensor in the Boyer-Lindquist coordinates  $\{t, r, \theta, \phi\}$  reads

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2Mr}{\Sigma}\right), \quad g_{t\phi} = -\frac{2aMr \sin^2 \theta}{\Sigma}, \quad g_{\phi\phi} = \frac{(\varpi^4 - a^2 \Delta \sin^2 \theta) \sin^2 \theta}{\Sigma}, \\ g_{\theta\theta} &= \Sigma, \quad g_{rr} = \frac{\Sigma}{\Delta}, \end{aligned} \quad (9)$$

with

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = \varpi^2 - 2Mr, \quad \varpi^2 = r^2 + a^2, \quad (10)$$

where  $M$  is the mass of the black hole and  $a$  corresponds to its angular momentum per mass  $M$ . By setting  $a = 0$  in Kerr's metric tensor, we recover the spherically symmetric Schwarzschild black hole metric tensor. To simplify some computations in Kerr (Carter, 1968) we can employ the following tetrad field

$$\begin{aligned} e_\mu^{(0)} &= \left( \sqrt{\frac{\Delta}{\Sigma}}, 0, 0, -a \sin^2 \theta \sqrt{\frac{\Delta}{\Sigma}} \right), \quad e_\mu^{(1)} = \left( 0, \sqrt{\frac{\Sigma}{\Delta}}, 0, 0 \right), \\ e_\mu^{(2)} &= (0, 0, \sqrt{\Sigma}, 0), \quad e_\mu^{(3)} = \left( \frac{-a \sin \theta}{\sqrt{\Sigma}}, 0, 0, \frac{r^2 + a^2}{\sqrt{\Sigma}} \sin \theta \right). \end{aligned} \quad (11)$$

Having briefed the basic concepts and notions for our analysis. We will start our investigation with the simpler Schwarzschild case and then move to the more complex Kerr one. In both of these investigations, we make the assumption that the dipole and the quadrupole contribution vanish independently. To address a more realistic scenario, this assumption should be dropped. However, we believe that it allows an interesting insight into the problem. In the framework provided by this assumption, we can use the results of Mukherjee et al. (2022) for the pole-dipole case and then deal with the quadrupole contribution to the MPD equations. For the quadrupole contribution, we first tackle the issue of whether the hidden momentum vanishes, and then we explore under which condition a geodesic motion is possible.

## 2 RADIAL MOTION IN THE SCHWARZSCHILD BACKGROUND

In the pole-dipole approximation, an extended body can move on a geodesic trajectory in the Schwarzschild background if it follows a radial trajectory (Mukherjee et al., 2022). The geodesic motion in this spherically symmetric background can be restricted to the equatorial plane, i.e.,

$$\mathcal{U}_t = -\tilde{E}_g, \quad \mathcal{U}_r = \pm \sqrt{\tilde{E}_g^2 - f \left(1 + \frac{\tilde{L}_z^2}{r^2}\right)} \left(1 - \frac{2M}{r}\right), \quad \mathcal{U}_\theta = 0, \quad \mathcal{U}_\phi = \tilde{L}_z, \quad (12)$$

where  $\tilde{E}_g$  and  $\tilde{L}_z$  denote the conserved specific geodesic energy and orbital angular momentum, respectively. It is obvious that for the radial motion  $\tilde{L}_z = 0$ . Because of the OKS SSC, i.e.  $\dot{w}^\mu = 0$ , the radial motion implies for the polar component of  $w^\mu$  that

$$\frac{dw^\theta}{d\tau} + \frac{\mathcal{U}^r}{r} w^\theta = 0, \quad (13)$$

i.e.  $rw^\theta = \text{constant}$ . By assuming for simplicity that this constant is zero, we get  $w^\theta = 0$ . On a radial trajectory, the azimuthal component of  $w^\mu$  obeys a similar equation to Eq. (13), which allows us also to have  $w^\phi = 0$  for simplicity.

Let us now discuss whether the hidden momentum vanishes for the radial trajectory, i.e. if the Eq. (6) is satisfied. By applying  $w^\theta = 0$  on Eq. (3) for the OKS SSC we get

$$\begin{aligned} K^{tr} &= K^{\theta\phi} = 0, \\ K^{t\theta} &= \frac{9M}{2r^2(2M-r^2)} \left\{ w^r (Q^{\theta r} w^t - Q^{\theta t} w^r) + r(2M-r) w^\phi (Q^{\theta\phi} w^t - Q^{\theta t} w^\phi) \right\}, \\ K^{t\phi} &= \frac{9M}{2r^2(2M-r^2)} \left\{ w^r (Q^{\phi r} w^t - Q^{\phi t} w^r) + w^\phi (Q^{rt} w^r - Q^{rr} w^t) \right. \\ &\quad \left. + r(r-2M) Q^{\theta\theta} w^t w^\phi \right\}, \\ K^{r\theta} &= \frac{9M}{2r^4} \left\{ r^3 w^\phi (Q^{\theta r} w^\phi - Q^{\theta\phi} w^r) + (2M-r) w^t (Q^{\theta r} w^t - Q^{\theta t} w^r) \right\}, \\ K^{r\phi} &= \frac{9M(2M-r)}{2r^4} \left\{ w^t (Q^{\phi r} w^t - Q^{\phi t} w^r) + w^\phi (Q^{tt} w^r - Q^{tr} w^t) \right\} - \frac{9Mr^3 Q^{\theta\theta} w^r w^\phi}{2r^4}. \end{aligned} \quad (14)$$

Taking the above relations into account when calculating  $K^{\mu\nu} w_\nu$  does not allow the contraction to vanish identically. Only if we assume  $w^\phi = 0$  along with  $w^\theta = 0$  leads to  $K^{\mu\nu} w_\nu = 0$ . Once the latter holds, we have a vanishing hidden momentum, allowing  $P^\mu \propto \mathcal{U}^\mu$ . Since we want the body to be on a geodesic trajectory, i.e.  $\dot{\mathcal{U}}^\mu = 0$ , then  $\dot{P}^\mu = \dot{m} \mathcal{U}^\mu$ . However, as we show below, we can achieve the same result without setting  $w^\phi = 0$  under our assumption of independently vanishing dipole and quadrupole terms.

The MPD equations in the case of geodesic motion reduce to

$$\dot{m} \mathcal{U}^\mu = -\frac{1}{2} R^\mu{}_{\nu\kappa\lambda} \mathcal{U}^\nu S^{\kappa\lambda} + F^\mu, \quad (15)$$

$$\dot{S}^{\mu\nu} = K^{\mu\nu}, \quad (16)$$

where  $F^\mu = -1/6 J^{\alpha\beta\gamma\delta} \nabla^\mu R_{\alpha\beta\gamma\delta}$ . Since on the radial trajectory for the pole-dipole the spin curvature coupling vanishes and  $\dot{S}^{\mu\nu} = 0$  (Mukherjee et al., 2022), then we have that: a)  $\dot{S}^{\mu\nu} = 0$  implies because of Eq. (16) that  $K^{\mu\nu} = 0$  and b)

$$\dot{m} \mathcal{U}^\mu = F^\mu. \quad (17)$$

We see that assuming independent vanishing of the dipole and quadrupole contributions, we are led to  $K^{\mu\nu} w_\nu = 0$  without necessarily setting  $w^\phi = 0$ .

For  $K^{\mu\nu} = 0$ , under the reasonable assumptions that in general  $w^r \neq 0$  and  $w^t \neq 0$ , Eq. (14) boils down to

$$\begin{aligned} Q^{tr} &= \frac{1}{2w^t w^r} \left( Q^{rr} [w^t]^2 + Q^{tt} [w^r]^2 \right) + \frac{rQ^{\theta\theta}}{2(r-2M)w^t w^r} \left\{ -(r-2M)^2 [w^t]^2 + r^2 [w^r]^2 \right\}, \\ Q^{t\theta} &= \frac{Q^{r\theta} w^t}{w^r}, \quad Q^{\theta\phi} = \frac{Q^{r\theta} w^\phi}{w^r}, \\ Q^{t\phi} &= \frac{Q^{r\phi} w^t}{w^r} + \frac{w^\phi}{2w^t [w^r]^2} \left\{ Q^{tt} [w^r]^2 - Q^{rr} [w^t]^2 - \frac{rQ^{\theta\theta} [r^2 (w^r)^2 + (2M-r)^2 (w^t)^2]}{2M-r} \right\}. \end{aligned} \quad (18)$$

By substituting the above relations Eq. (18) into  $F^\mu$ , we obtain

$$\begin{aligned} F^r &= \frac{3M [r^2 (w^r)^2 - (r-2M)^2 (w^t)^2]}{2r^4 (w^r)^2} \left[ \left\{ Q^{rr} + (2M-r) Q^{\theta\theta} \right\} (w^\phi)^2 \right. \\ &\quad \left. + (Q^{\phi\phi} - Q^{\theta\theta}) (w^r)^2 - 2Q^{r\phi} w^r w^\phi \right], \end{aligned} \quad (19)$$

and  $F^t = F^\theta = F^\phi = 0$ . The latter implies that  $\dot{m} = 0$ , which in turn implies that  $F^r = 0$  resulting in the additional constraint

$$Q^{\phi\phi} = \frac{1}{[w^r]^2} \left[ Q^{\theta\theta} \left( [w^r]^2 + (r-2M) [w^\phi]^2 \right) + w^\phi \left( 2Q^{r\phi} w^r - Q^{rr} w^\phi \right) \right]. \quad (20)$$

To sum up, we have imposed geodesic motion in the Schwarzschild spacetime to a pole-dipole-quadrupole body under OKS SSC, assuming that only the mass quadrupole contributes to the quadrupole tensor. Moreover, we have assumed that the dipole and the quadrupole contributions to the MPD vanish independently. This setup led us to the constraints Eq. (18) and Eq. (20), which are coupling components of the mass quadrupole tensor  $Q^{\mu\nu}$  and the components of the reference vector  $w^\mu$ . The MPD equations do not prescribe an evolution equation for the quadrupole tensor and so from this point of view, there is no evolution equation for the mass quadrupole tensor. However, for the reference vector, we have  $w^{\dot{\mu}} = 0$ , hence the constraints Eq. (18) and Eq. (20) dictate the way the mass quadrupole tensor evolves under OKS SSC, if the pole-dipole-quadrupole body is set to follow a geodesic trajectory.

## 2.1 Equatorial motion in the Kerr background

In this section, we discuss under which conditions a pole-dipole-quadrupole body can follow an equatorial geodesic trajectory in the Kerr background. As in Section-(2), we discuss the case for which the mass quadrupole is the only component of the quadrupole moment (Eq. (3)). On the equatorial plane, Eq. (13) holds also for Kerr (Harms et al., 2016). Hence, we can set  $w^\theta = 0$  for simplicity, which in turn implies that  $w^{(\theta)} = 0$ . The  $K^{\mu\nu}$  expressed in

the tetrad field frame reads

$$\begin{aligned}
K^{(t)(r)} &= K^{(\theta)(\phi)} = 0, \\
K^{(t)(\theta)} &= \frac{9M}{2r^3} \left\{ w^{(r)} \left( Q^{(t)(\theta)} w^{(r)} - Q^{(r)(\theta)} w^{(t)} \right) + w^{(\phi)} \left( Q^{(\theta)(\phi)} w^{(t)} - Q^{(\theta)(t)} w^{(\phi)} \right) \right\}, \\
K^{(t)(\phi)} &= -\frac{9M}{2r^3} \left\{ w^{(r)} \left( Q^{(r)(\phi)} w^{(t)} - Q^{(t)(\phi)} w^{(r)} \right) + w^{(\phi)} \left( Q^{(t)(r)} w^{(r)} - Q^{(r)(r)} w^{(t)} \right) \right. \\
&\quad \left. + Q^{(\theta)(\theta)} w^{(t)} w^{(\phi)} \right\}, \\
K^{(r)(\theta)} &= \frac{9M}{2r^3} \left\{ w^{(t)} \left( Q^{(t)(\theta)} w^{(r)} - Q^{(r)(\theta)} w^{(t)} \right) + w^{(\phi)} \left( Q^{(\theta)(\phi)} w^{(r)} - Q^{(r)(\theta)} w^{(\phi)} \right) \right\}, \\
K^{(r)(\phi)} &= -\frac{9M}{2r^3} \left\{ w^{(t)} \left( Q^{(r)(\phi)} w^{(t)} - Q^{(t)(\phi)} w^{(r)} \right) + w^{(\phi)} \left( Q^{(t)(t)} w^{(r)} - Q^{(t)(r)} w^{(t)} \right) \right. \\
&\quad \left. + Q^{(\theta)(\theta)} w^{(r)} w^{(\phi)} \right\}.
\end{aligned} \tag{21}$$

The above equations imply that on the equatorial plane  $K^{(\mu\nu)}w_{(\nu)}$  does not vanish if we just set  $w^{(\theta)} = 0$ , we need to set also  $w^{(\phi)} = 0$  to achieve it. However, since [Mukherjee et al. \(2022\)](#) showed that the spin curvature coupling for a pole-dipole body vanishes for equatorial trajectories satisfying the relation  $\tilde{L}_z = a\tilde{E}_g$ , by following the same procedure as in [Section-\(2\)](#), i.e. by assuming that the dipole and quadrupole contribution vanishes independently, we end up with  $K^{(\mu\nu)} = 0$ . This results in  $K^{(\mu\nu)}w_{(\nu)} = 0$  and [Eq. \(21\)](#) provides the following relations between the components of the quadrupole moment tensor:

$$\begin{aligned}
Q^{(t)(r)} &= \frac{Q^{(r)(r)} [w^{(t)}]^2 + Q^{(t)(t)} [w^{(r)}]^2 + Q^{(\theta)(\theta)} \left( -[w^{(t)}]^2 + [w^{(r)}]^2 \right)}{2w^{(t)}w^{(r)}}, \\
Q^{(t)(\theta)} &= \frac{Q^{(r)(\theta)} w^{(t)}}{w^{(r)}}, \quad Q^{(\theta)(\phi)} = \frac{Q^{(r)(\theta)} w^{(\phi)}}{w^{(r)}}, \\
Q^{(t)(\phi)} &= \frac{Q^{(r)(\phi)} w^{(t)}}{w^{(r)}} + \frac{w^{(\phi)}}{2w^{(t)}[w^{(r)}]^2} \left\{ Q^{(t)(t)} [w^{(r)}]^2 - Q^{(r)(r)} [w^{(t)}]^2 \right. \\
&\quad \left. + Q^{(\theta)(\theta)} \left( [w^{(t)}]^2 + [w^{(r)}]^2 \right) \right\}.
\end{aligned} \tag{22}$$

If we now substitute the above relations into  $F^{(\mu)}$ , we find that

$$\begin{aligned}
F^{(r)} &= \frac{9M\sqrt{\Delta}}{r^5 [w^{(r)}]^2} \left( [w^{(t)}]^2 - [w^{(r)}]^2 \right) \left[ Q^{(\theta)(\theta)} \left( [w^{(r)}]^2 + [w^{(\phi)}]^2 \right) \right. \\
&\quad \left. + w^{(\phi)} \left( 2Q^{(r)(\phi)} w^{(r)} - Q^{(r)(r)} w^{(\phi)} \right) - Q^{(\phi)(\phi)} [w^{(r)}]^2 \right]
\end{aligned}$$

and  $F^{(t)} = F^{(\theta)} = F^{(\phi)} = 0$ . The latter result implies because of [Eq. \(17\)](#) that  $m = 0$  and hence, from  $F^{(r)} = 0$ , we obtain the following relation

$$Q^{(\phi)(\phi)} = \frac{1}{[w^{(r)}]^2} \left[ Q^{(\theta)(\theta)} \left( [w^{(r)}]^2 + [w^{(\phi)}]^2 \right) + w^{(\phi)} \left( 2Q^{(r)(\phi)} w^{(r)} - Q^{(r)(r)} w^{(\phi)} \right) \right]. \tag{23}$$

The interpretation we give to the constraints Eq. (22) is the same as in Section-(2). Namely, in order for the pole-dipole-quadrupole body to follow a geodesic trajectory the parallel transported along the geodesic reference vector  $w^{(\mu)}$  has to dictate the evolution of the mass quadrupole through these constraints.

### 3 CONCLUSIONS

Using trajectories of the pole-dipole body that coincide with geodesics on a black hole background, we explored under which condition a pole-dipole-quadrupole body obeying OKS SSC can follow these trajectories as well when the mass quadrupole tensor is the only component of the quadrupole tensor. To achieve these conditions, we have used the not-so-realistic assumption that the dipole and quadrupole contributions to the MPD vanish independently. We showed that there are constraints between the components of the mass quadrupole tensor and the reference vector  $w^\mu$ , which have to be obeyed in order for the body to follow the aforementioned trajectories. Since the reference vector is parallel transported along a geodesic, these constraints, in a sense, substitute the missing evolution equations for the quadrupole moment from the MPD equations.

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